§5.2: Series Solutions Near an Ordinary Point, Part I

Last chapter we studied methods of solving linear SODE with constant coefficients. Now we turn to eqn's with coefficients that are functions of the independent variable:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\,y = 0 \qquad (*)$$

We will consider the homogeneous eqn first (maybe not ever the non-homog one).

Some examples of problems that we will solve are

Bessel's eqn: $x^2y'' + xy' + (x^2-\nu^2)y = 0$, and

Legendre's eqn: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$.

Thus, we shall primarily consider the case when P, Q, R are polynomials, but in general we can consider P, Q, R to be general analytic functions (representable as power series).

The sol'n of the eqn $(*)$ is closely related to the behavior of $P(x)$ at (or near) a point x_0.

We want to solve $(*)$ in an interval containing the point x_0.

A point x_0 such that $P(x_0) \neq 0$ is called an ordinary point.

Since P is continuous, there is an interval about x_0 in which $P(x)$ is never 0.

In that interval, we can divide by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0 \qquad (**),$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$.

Then, by the FEUT, there exists a unique sol'n of eq'n (*) that also satisfies the initial condition $y(x_0) = y_0$, and $y'(x_0) = y_0'$.

Now we study ~~the~~ such sol'ns in ~~the~~ a neighborhood of an ordinary point.

If $P(x_0) = 0$, then x_0 is called a singular point of (*). In this case, at least one of $Q(x)$ or $R(x)$ is nonzero, in which case one of $p(x)$, $q(x)$ becomes unbounded as $x \to x_0$. Thus the FEUT does not apply.

We will study singular points only briefly in 5.4.

→ In a nbd of an ordinary pt, we try to find sol'ns of the form

$$y = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

and assume that the series converges in the interval $|x-x_0| < R$ for some $R > 0$.

The most practical way to determine the coefficients a_n is to substitute the series and its derivatives in for y, y', y''.

Ex. $y'' + y = 0 \qquad -\infty < x < \infty$ $\qquad P(x)=1,\ Q(x)=0,\ R(x)=1$, hence every pt is ordinary. Choose $x_0=0$.

We already know that $\sin x$ and $\cos x$ are the fundamental solutions of this eqn, so this new method is not necessary. However, it does nicely illustrate the method.

Assume $y = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$ is the sol'n and is convergent in some interval $|x| < R$.

Differentiating term by term we get:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = 2a_2 + 6a_3 x + \dots + n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in, we get:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to combine this into one sum: we need to shift the index on the first one.

let $m = n-2 \Rightarrow n = m+2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

Put back $m=n$:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n$$

For this eq'n to be satisfied for all x, the coefficient of each power of x must be 0:

$$(n+2)(n+1)a_{n+2} + a_n = 0 \qquad n = 0, 1, 2, \ldots \qquad (\ast\ast\ast)$$

This is referred to as a recurrence relation.

This eq'n relates each term to the one that came two steps before it. Thus we may determine the evens and odds separately.

For the evens, we get:

$$a_2 = -\frac{a_0}{2\cdot 1} = \frac{-a_0}{2!}, \quad a_4 = \frac{-a_2}{4\cdot 3} = \frac{+a_0}{4!}, \quad a_6 = \frac{-a_4}{6\cdot 5} = \frac{-a_0}{6!}$$

In general, if $n = 2k$, then $a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \qquad k = 1, 2, 3, \ldots$

Similarly, for odd integers $n = 2k+1$ we have $a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1, \quad k = 1, 2, 3, \ldots$

Substituting back into the eq'n we have

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - + + + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\right]$$

$$= \underbrace{a_0}_{c_1} \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{y_1} + \underbrace{a_1}_{c_2} \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{y_2}$$

Thus, we have formally found 2 sol'ns.

RE. Check the radii of convergence of the sol'ns and notice that these are the series for cosine (even) and sine (odd). (Taylor)

Thus, the sol'n is $y = a_0 \cos x + a_1 \sin x$.
a_1 and a_0 are unknown constants that can be solved for using initial data.

Suppose that we didn't know these series represent cosine and sine. We can ask whether these two series sol'ns

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

are linearly independent. (do they form a fund. sol'n set?)

It's easy to see that $C(0) = 1$ and $S(0) = 0$ (where $0 = x_0$ in this example). Moreover, $C'(x) = -S(x)$ and $S'(x) = C(x)$ for all x. In particular, $C'(0) = 0$ and $S'(0) = 1$

Then $W(C,S)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

So C and S do form a fundamental sol'n set.

We can now use this DE to define sine and cosine:
$y = \sin(x)$ is the unique sol'n to the SODE $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$
$y = \cos(x)$ is the unique sol'n to the SODE $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

COOL!

$(r^2+1)(r^2-4)=0$

$r^2(r^2-4)+1(r^2-4)=0$

$r^4-4r^2+r^2-4=0$

__Ex.__ $y^{(4)}-3y''+4y=0$

__sol'ns are:__ $y_1=\cos t,\ y_2=\sin t,\ y_3=e^{2t},\ y_4=te^{2t}$

__Wronskian sucks:__ ew.

general sol'n is $y=c_1\cos t+c_2\sin t+c_3e^{2t}+c_4te^{2t}$.

__Ex.__ $y^{(4)}-3y''+4y=8e^{2t}+\cos t$

__Guess:__ $Y(t)=At^2e^{2t}+Bt\cos t+Ct\sin t$

Go from there.

__Ex.__ $y^{(4)}+y'''=\sin(2t)$

$r^4-r^3=0$

$r^3(r-1)=0$

e^t

$y'''=0$

$y=c_1+c_2t+c_3t^2$

$y=c_1+c_2t+c_3t^2+c_4e^t$

__Guess:__ $Y=A\sin 2t+B\cos 2t$

Plug in. Solve for A, B.

Find the first 5 nonzero terms in the solution of

$$\underbrace{y'' - xy' - y = 0}_{\text{solve this for } x_0 = 0.}, \quad y(0) = 2, \quad y'(0) = 1$$

Ex. Airy's Equation: $y'' - xy = 0 \qquad -\infty < x < \infty$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$ since $x_0 = 0$ is an ordinary pt.

then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Plug in:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$m = n-2 ! \qquad \cancel{m = n+1}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

Thus we get $(n+2)(n+1) a_{n+2} = a_{n-1} \qquad n = 1, 2, 3, \dots$

We also must put $a_2 = 0$, hence also $a_5 = a_8 = a_{11} = \dots = 0$.

When $n = 1$, we get $a_3 = \frac{1}{3 \cdot 2} a_0$

$n = 4$, $\qquad a_6 = \frac{1}{6 \cdot 5} a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0$

$n = 7$, $\qquad a_9 = \frac{1}{9 \cdot 8} a_6 = \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} a_0$

We get a general formula $a_{3n} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n-1)(3n)} a_0$

Similarly, starting with $n=2$, we get

$$a_{3n+1} = \frac{1}{3\cdot4\cdot6\cdot7\cdots(3n)(3n+1)}\, a_1$$

Thus, the general solution to Airy's eq'n looks like:

$$y = a_0\left[1 + \frac{x^3}{2\cdot3} + \frac{x^6}{2\cdot3\cdot5\cdot6} + \frac{x^9}{2\cdot3\cdot5\cdot6\cdot8\cdot9} + \cdots\right]$$
$$+ a_1\left[x + \frac{x^4}{3\cdot4} + \frac{x^7}{3\cdot4\cdot6\cdot7} + \frac{x^{10}}{3\cdot4\cdot6\cdot7\cdot9\cdot10} + \cdots\right]$$

<u>RE</u>. Show that the radius of convergence for each of these sol'ns is ∞.

(*) <u>On the Elmo</u>: Show the pics on pg. 257 of the polynomial approximations of these sol'ns.

<u>Ex</u>. We solved the DE: $y'' - xy' - y = 0$ w/ $x_0 = 0$ and got

$$y = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!}}_{\text{evens}} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{2^n n!\, x^{2n+1}}{(2n+1)!}}_{\text{odds.}}$$

Now we want to solve the IVP:

$$y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = 1$$

$y(0) = a_0$

$y'(0) = a_1$

$$\Rightarrow \quad y(x) = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + 1 \sum_{n=0}^{\infty} \frac{2^n n! \, x^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \quad y_{(4)}(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$$

Plot this and estimate the interval on which it is "relatively accurate".