Project 7 will cover Chapter 4.

Now on to Chapter 5:

Ch.5: Series Solutions of Second Order Linear DEs.

5.1: Review of Power Series

Some rapid-fire review:

1. A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

where a_n are constants and x_0 is the base point or center of the series; x is a variable.
A power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is said to converge at a pt x iff

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

exists for that x. This series always converges for $x=x_0$. It may converge for all other x, or it may not.

2. The series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is called absolutely convergent at x iff

$$\sum_{n=0}^{\infty} |a_n (x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

converges at x.

If a series is absolutely convergent, then it is convergent (at a pt x).

3. The <u>Ratio Test</u>: If $a_n \neq 0$, $x =$ fixed, then we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L.$$

Then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is absolutely conv. if $|x-x_0|L<1$, diverges if $|x-x_0|L>1$, and is inconclusive if $|x-x_0|L=1$ for that value of x.

<u>Ex.</u> For which values of x does

$$\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n \qquad \text{converge?}$$

Use ratio test to get $|x-2|<1$

So it converges for $1<x<3$.

We need to check the end points $x=1$ and $x=3$.

Plug back in to get:

$$x=1: \quad \sum_{n=1}^{\infty} (-1)^{n+1} n (1-2)^n = \sum_{n=1}^{\infty} (-1)^{n+1} n (-1) = \sum_{n=1}^{\infty} (-1)^n n$$

which is div.

Similarly for $x=3$.

So the interval of convergence is $x \in (1,3)$.

Radius of conv. is 1.

Ex. Determine the radius of convergence:

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 2^n}$$

Ratio test yields: $\frac{|x+1|}{2} < 1$

or $|x+1| < 2$ Thus, $R = 2$.

The interval of conv. is $-3 < x < 1$ But what about the endpoints?

Plug in: $x = -3$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{n 2^n} =$

$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ conv. by alt. series test

$x = 1$: $\sum_{n=1}^{\infty} \frac{(2^n)}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (harmonic series)

So the IC is $x \in [-3, 1)$.

Ex. Mult. and addition of power series preserve the radius of convergence, but maybe not the endpoints. Division may shrink the radius of convergence.

Ex. Taylor Series. Let f be a smooth function with derivatives of all orders at $|x - x_0| < R$.

The Taylor series expansion of f at x_0 is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Ex. Write the Taylor Series expansions for $y = e^x$, $y = \sin x$, $y = \cos x$ at $x = 0$.

$$\begin{cases} e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty\ ! \\[2ex] \sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\[2ex] \cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{cases}$$

ex. Shift the index: $\displaystyle\sum_{n=2}^{\infty} a_n x^n$ so that it starts at 0.

let $m = n-2$

$$\boxed{\sum_{m=0}^{\infty} a_{m+2} x^{m+2}}$$

Ex. Shift ~~the other way~~: $\displaystyle\sum_{n=2}^{\infty} (n+2)(n+1)(x-x_0)^{n-2} a_n$

get $\displaystyle\sum_{n=0}^{\infty} (n+4)(n+3)\, a_{n+2} (x-x_0)^n$

Ex. differentiating and integrating power series is straight forward.

recall $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \dfrac{1}{1-r} = \displaystyle\sum_{n=0}^{\infty} r^n$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Integrate $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ or something.

Ex. x^2 at $x_0 = -1$ Taylor series.

Ex. Find P.S. for $\frac{1}{(1-x)^2}$ at $x_0 = 0$.

$$\left(\frac{1}{(1-x)}\right)' = + \frac{1}{(1-x)^2} \quad \text{yup.}$$