3.6: Variation of Parameters

To close out chapter 6, we study another method of solving nonhomog. linear SODE.

The method is called Variation of Parameters, and its main appeal is that it is a general method: In principle (at least) it can be applied to any SODE, and doesn't require us to guess sol'ns.

Ex. $y'' + 4y = 3\csc t$ $\quad(*)$

This is not a good candidate for the method of Undetermined Coefficients since the RHS $g(t) = 3\csc t$ involves a quotient of $\sin t, \cos t$ rather than a product. We also don't have any good guesses as to what function might satisfy something like this.

First, solve the homog. problem:

$$y'' + 4y = 0$$

we get general sol'n:

$$y = C_1 \cos 2t + C_2 \sin 2t$$

The basic idea of variation of parameters is to let the constants (parameters) C_1 and C_2 vary in t:

$$Y = u_1(t) \cos 2t + u_2(t) \sin 2t \qquad (**)$$

Now we try to determine functions u_1 and u_2 that make this expression a sol'n of the nonhomog. problem.

If we just substitute this in for y in the original problem, then we will have <u>one</u> DE w/ <u>two</u> unknown functions. In other words, the system will be underdetermined, and there will be infinitely many solutions (if any).

Instead, we may try to pose additional constraints on u_1 and u_2. We'll see that it is possible to choose the second "constraint" so that the computation itself is fairly simple.

Take the derivative of (**):

$$Y' = u_1'(t)\cos 2t - 2u_1(t)\sin 2t + u_2'(t)\sin 2t + 2u_2(t)\cos 2t$$
$$= -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t$$

Let's suppose (for no apparent reason right now) that

$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0 \qquad (***)$$

So that $Y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t$

Differentiating (****) again, we get

$$Y''(t) = -4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t$$

Now substitute these formulae for Y and Y'' into $(*)$ to get:

$$-\cancel{4u_1(t)\cos 2t} - \cancel{4u_2(t)\sin 2t} - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t + \cancel{4u_1(t)\cos 2t}$$
$$+4\cancel{u_2(t)\sin 2t} = 3\csc t$$

or $$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t \qquad (4*)$$

This is a second eq'n relating u_1', u_2'.

Now we have a system of 2 eq'ns $(3*)$ and $(4*)$ w/ 2 unknowns u_1' and u_2'.

$$\begin{cases} u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0 \\ -2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t \end{cases}$$

This system must be satisfied for all t in the interval of the sol'n. (This is something to think about when solving an IVP.)

$$\begin{array}{r} 2u_1'(t)\cos 2t\sin 2t + 2u_2'(t)\sin^2 2t = 0 \\ -2u_1'(t)\cos 2t\sin 2t + 2u_2'(t)\cos^2 2t = 3\csc t\cos 2t \\ \hline 2u_2'(t) = 3\dfrac{\cos 2t}{\sin t} \end{array}$$

or $$u_2'(t) = \frac{3\cos 2t}{2\sin t}$$

$$u_1'(t) = -u_2'(t)\frac{\sin 2t}{\cos 2t} = \frac{-3\cancel{\cos 2t}}{2\sin t}\frac{\sin 2t}{\cancel{\cos 2t}} = \frac{-3\cdot\cancel{2\sin t}\cos t}{\cancel{2\sin t}}$$

$$u_1'(t) = -3\cos t$$

We can rewrite $u_2'(t)$ as $\dfrac{3}{2}\dfrac{\cos 2t}{\sin t} = \dfrac{3(1-2\sin^2 t)}{2\sin t} = \dfrac{3}{2}\csc t - 3\sin t$

Thus,

$$u_1 = -3\sin t + c_1 \quad \text{and}$$
$$u_2 = \tfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2$$

and

$$Y(t) = -3\sin t\cos 2t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3\cos t\sin 2t$$
$$\underbrace{+\,c_1\cos 2t + c_2\sin 2t}_{\text{these were the homog. eqn's sol'ns !!!}}$$

Using some more half-angle formulas we can rewrite this as:

$$y = \underbrace{3\sin t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t}_{Y(t)} + c_1\cos 2t + c_2\sin 2t$$

Even in this seemingly "simple" example, this method is a bit convoluted.

Let's recap everything "symbolically" (i.e., in general):

Variation of Parameters:

It's just Cramer's Rule!

get $u_1' = \dfrac{-y_2(t)\, g(t)}{W(y_1,y_2)(t)}$ $\quad u_2' = \dfrac{y_1(t)\, g(t)}{W(y_1,y_2)(t)}$

so $u_1 = -\displaystyle\int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(y_1,y_2)(s)}\, ds$ and

$u_2 = \displaystyle\int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1,y_2)(s)}\, ds$

Sol'n is
$Y = u_1 y_1 + u_2 y_2$.

2. $y'' - y' - 2y = 2e^{-t}$
5. $y'' + y = \tan t \quad 0<t<\pi/2$
10. $y'' - 2y' + y = e^t/(1+t^2)$

13. $t^2 y'' - 2y = 3t^2 - 1, \ t>0 \quad y_1 = t^2 \quad y_2 = t^{-1}$

17. $x^2 y'' - 3xy' + 4y = x^2 \ln x \quad 0<t<1 \quad y_1 = e^t \quad y_2 = t$