3.5: Nonhomogeneous Equations; Undetermined Coefficients

Now, back to the general equation:

$$\mathcal{L}[y] = y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

where, p, q, g are continuous on some interval I.

The eq'n

$$\mathcal{L}[y] = y'' + p(t)y' + q(t)y = 0 \qquad (**)$$

is called the homogeneous eq'n corresponding to $(*)$

Thm 1. If Y_1 and Y_2 are two sol'ns of the nonhomog. eq'n $(*)$, then their difference $Y_1 - Y_2$ is a sol'n of the homog. eq'n $(**)$. In addition, if y_1 and y_2 form a fundamental sol'n set of (for) $(**)$, then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

for certain constants c_1 and c_2.

Pf | $\mathcal{L}[Y_1] = g(t)$, $\mathcal{L}[Y_2] = g(t)$

$$\mathcal{L}[Y_1 - Y_2] = \mathcal{L}[Y_1] - \mathcal{L}[Y_2] = g(t) - g(t) = 0. \quad \blacksquare$$

verify this!

Thm 2. The general sol'n of the nonhomog. eq'n $(*)$ can be written as

$$y(t) = \varphi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$

where y_1, y_2 are a fund. sol'n set of $(**)$, ~~and~~ c_1, c_2 are constants, and Y is a specific sol'n of $(*)$.

This says that sol'ns of $(*)$ may differ by sol'ns of $(**)$.

Consequently, we see that we could solve (*) by first finding the solńs to (**), then finding the particular solń to (*) and adding the solńs together.

Method of Undetermined Coefficients:

Idea: Make a guess as to the form of the solń, plug it into the DE, and solve for the coefficients.

Example: $y'' - 3y' - 4y = 3e^{2t}$

Guess: solń should look like $y = Ae^{2t}$ for some constant A.

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

Plugging in: $4A\cancel{e^{2t}} - 6A\cancel{e^{2t}} - 4A\cancel{e^{2t}} = 3\cancel{e^{2t}}$

$$-6A = 3$$

$$A = -\tfrac{1}{2}$$

So the solń is $y = -\frac{1}{2}e^{2t}$

But this is just the particular solń. We can form other solńs by adding solńs of the homog. eqn

$$y'' - 3y' - 4y = 0$$

Char. Eqn: $r^2 - 3r - 4 = 0$

$$(r-4)(r+1) = 0$$

$$r = 4, -1$$

$$y = C_1e^{4t} + C_2e^{-t}$$

Thus, the general solń of the original DE is:

$$\boxed{y = C_1e^{4t} + C_2e^{-t} + -\tfrac{1}{2}e^{+2t}}$$

Ex. $y'' - 3y' - 4y = 2\sin t$ Find A particular sol'n.

Guess: $Y(t) = A\sin t + B\cos t$

$Y'(t) = A\cos t - B\sin t$

$Y''(t) = -A\sin t - B\cos t$

The DE is:

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t$$

$$\sin t(-A + 3B - 4A) + \cos t(-B - 3A - 4B) = 2\sin t$$

get 2 eq'ns: $-5A + 3B = 2$ $\Rightarrow$ $-15A + 9B = 6$

$-3A - 5B = 0$ $\quad$ $-15A - 25B = 0$

$34B = 6$

$B = 6/34 = 3/17$

$\Rightarrow -3A = 5(3/17)$

$A = -5/17$

so the sol'n is $\boxed{Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t}$

RE: Find the general sol'n.

Ex. $y'' - 3y' - 4y = -8e^t\cos 2t$

Guess: $Y(t) = Ae^t\cos 2t + Be^t\sin 2t$

This will be a mess... do it.

get: $\boxed{Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t}$

Ex. $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t$

Add the sol'ns of the previous examples:

$$\boxed{Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t}$$

Ex. $y'' - 3y' - 4y = 2e^{-t}$

Guess:
$$\left.\begin{aligned} Y(t) &= Ae^{-t} \\ Y'(t) &= -Ae^{-t} \\ Y''(t) &= Ae^{-t} \end{aligned}\right\} \quad \begin{aligned} &Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t} \\ &\Rightarrow \quad 0A = 2 \quad \times \\ &\text{impossible!} \end{aligned}$$

What went wrong!? $Y(t) = Ae^{-t}$ is a sol'n of the homog. eq'n!
Where do we go from here?

Either: 1. use a different method, or

2. make a better guess.

Consider the eq'n: $y' + y = 2e^{-t}$

This has the same problem, but since it is first order, we know how to solve it:

$$\left.\begin{aligned} p(t) &= 1 \\ g(t) &= 2e^{-t} \end{aligned}\right\} \Rightarrow \mu(t) = e^{t}$$

The sol'n is:

$$y(t) = e^{-t}\int_{t_0}^{t} e^{s}\, 2e^{-s}\, ds + Ce^{-t}$$

$$= \underbrace{2te^{-t}}_{\text{New!}} + \underbrace{Ce^{-t}}_{\text{we knew}}$$

So we guess that Ate^{-t} is a sol'n of the first DE, by analogy.

$$Y(t) = Ate^{-t}$$
$$Y'(t) = Ae^{-t} - Ate^{-t}$$
$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$$

Plugging in:

$$-2Ae^{-t} + \cancel{Ate^{-t}} - 3Ae^{-t} + 3\cancel{Ate^{-t}} - \cancel{4Ate^{-t}} = 2e^{-t}$$

$$-5A = 2 \Rightarrow A = -2/5$$

So $\boxed{Y(t) = \frac{-2}{5} t e^{-t}}$

Based on these examples, we can make a table to help us guess:

The particular sol'n of the DE: $ay'' + by' + cy = g_i(t)$ has the form:

Table 3.5.1.

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 + a_1 t^1 + a_2 t^2 + \dots + a_n t^n$	$t^s(A_0 + A_1 t + A_2 t^2 + \dots + A_n t^n)$
$P_n(t) e^{\alpha t}$	$t^s(A_0 + A_1 t + \dots + A_n t^n) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s(A_0 + A_1 t + \dots + A_n t^n) e^{\alpha t} \cos \beta t + t^s(B_0 + B_1 t + B_2 t^2 + \dots + B_n t^n) e^{\alpha t} \sin \beta t$

where $s = 0, 1, 2$ is the smallest number that will ensure no term in $Y_i(t)$ will be a sol'n of the homog. eqn.