3.4. Repeated Roots; Reduction of Order

Back to the linear, homog., constant coeffient problem:

$$ay''+by'+cy=0 \qquad a\neq 0,\ b,c\in\mathbb{R}$$

This has characteristic eqn:

$$ar^2+br+c=0$$

If $b^2-4ac=0$, then this equation has only one (repeated) root,

$$r_1=r_2=\frac{-b}{2a}.$$

Based on previous sections, we know this yields

$$y=e^{-bt/2a} \quad \text{as a soln of the SODE.}$$

But both r's yield the same soln!

It's also not obvious how to find another soln without having another root.

Ex. $y''+4y'+4y=0$ So ~~all~~ one soln is $C_1e^{-2t}=y_1$.

$$r^2+4r+4=0$$
$$(r+2)^2=0$$
$$r=-2$$

To find the second soln, we use a method discovered by D'Alambert:

Let $v(t)$ be a function and assume $y_2=v(t)y_1=v(t)e^{-2t}$.

Then plug back into the DE:

$$y=ve^{-2t}$$
$$y'=-2ve^{-2t}+v'e^{-2t}$$
$$y''=4ve^{-2t}-2v'e^{-2t}-2v'e^{-2t}+v''e^{-2t}$$

Plugging into $y'' + 4y' + 4y = 0$ we get

$$\cancel{4ve^{-2t}} - \cancel{4v'e^{-2t}} + v''e^{-2t} - \cancel{8ve^{-2t}} + \cancel{4v'e^{-2t}} + \cancel{4ve^{-2t}} = 0$$

or just $v''e^{-2t} = 0$

e^{-2t} can never be 0, so we get the DE $v'' = 0$ for v.
Thus $v = C_2 t + C_3$, and

$$y_2 = C_2 t e^{-2t} + C_3 e^{-2t}$$

The general sol'n is: $y = C_1 y_1 + C_2 y_2 = \underline{C_1 e^{-2t}} + C_2 t e^{-2t} + \underline{C_3 e^{-2t}}$

or $\boxed{y = C_1 e^{-2t} + C_2 t e^{-2t}}$

Ex. Check that e^{-2t}, te^{-2t} form a fund sol'n set:

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (-2t+1)e^{-2t} \end{vmatrix} = e^{-4t}(-2t+1+2t) = e^{-4t} \neq 0.$$

We could repeat this process in general for the SODE
$$ay'' + by' + cy = 0$$
with root(s) $r_1 = r_2 = -b/2a$, but we won't (maybe on a project!?).

What we obtain is this: If $ay'' + by' + cy = 0$ has repeated root $r = -b/2a$, then the general sol'n is given by

$$\boxed{y = C_1 e^{-b/2a\, t} + C_2 t e^{-\frac{b}{2a} t}}$$

Ex. $y'' - y' + \frac{1}{4} y = 0$

$$r^2 - r + \frac{1}{4} = 0$$
$$\left(r - \frac{1}{2}\right)^2 = 0$$
$$r = \frac{1}{2}$$

$\Rightarrow$ sol'n: $\boxed{y = c_1 t e^{+\frac{1}{2}t} + c_2 e^{+\frac{1}{2}t}}$

Add initial data: $y(0) = 2$, $y'(0) = \frac{1}{3}$

$$y' = c_1 e^{+\frac{1}{2}t} + \frac{1}{2} c_1 t e^{+\frac{1}{2}t} + \frac{1}{2} c_2 e^{+\frac{1}{2}t}$$
$$y'(0) = c_1 + \frac{1}{2} c_2 = \frac{1}{3}$$
$$y(0) = c_2 = 2 \quad \Rightarrow \quad c_1 = \frac{-2}{3}$$

so the sol'n is: $\boxed{y = \frac{-2}{3} t e^{\frac{1}{2}t} + 2 e^{\frac{1}{2}t}}$

Ex. Reduction of Order in general.

Given that $y_1(t) = \frac{1}{t}$ is a sol'n of $2t^2 y'' + 3t y' - y = 0$, $t > 0$, Find another sol'n.

set $y_2(t) = \frac{v(t)}{t}$

$$y_2'(t) = \frac{t v' - v}{t^2} = \frac{1}{t} v' - \frac{1}{t^2} v$$
$$y_2''(t) = \frac{t v'' - v'}{t^2} - \frac{t^2 v' - 2t v}{t^4} = \frac{1}{t} v'' - \frac{1}{t^2} v' - \frac{1}{t^2} v' + \frac{2}{t^3} v$$
$$= \frac{1}{t} v'' - \frac{2}{t^2} v' + \frac{2}{t^3} v$$

The DE becomes:

$$2t^2\left(\frac{1}{t} v'' - \frac{2}{t^2} v' + \frac{2}{t^3} v\right) + 3t\left(\frac{1}{t} v' - \frac{1}{t^2} v\right) - \frac{1}{t} v = 0$$
$$2t v'' - 4v' + \cancel{\frac{4}{t} v} + 3v' - \cancel{\frac{3}{t} v} - \cancel{\frac{1}{t} v} = 0$$
$$2t v'' - v' = 0$$

This is a first order linear DE for v'!

$$2tv'' - v' = 0$$

$$\Rightarrow \quad 2t\,v'' = v' \quad \Longrightarrow \quad 2t\frac{dv'}{dt} = v'$$

$$2\frac{dv'}{v'} = \frac{1}{t}\,dt$$

$$2\ln v' = \ln t$$

$$(v')^2 = t$$

$$v' = \sqrt{t}$$

$$v = \frac{2}{3}t^{3/2} + k$$

so $y = \frac{1}{t}\left(\frac{2}{3}t^{3/2} + k\right) = \frac{2}{3}t^{1/2} + \frac{k}{t}$

but $\frac{k}{t}$ was already the sol'n y_1. Thus the general sol'n is

$$y = C_1 t^{1/2} + C_2 t^{-1}$$

as long as the Wronskian is nonzero.

$$W(t^{1/2}, t^{-1}) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2} \neq 0 \text{ for } t>0.$$

Ex. (27) $xy'' - y' + 4x^3y = 0$, $x>0$ $\qquad y_1(x) = \sin x^2$

Put $y = v\,y_1 = v\sin x^2$

Then $y' = 2x\cos x^2\,v + v'\sin x^2$

$y'' = (2xv)'\cos x^2 - 4x^2 v\sin x^2 + v''\sin x^2 + 2xv'\cos x^2$

The DE becomes: $\nwarrow$ $2v\cos x^2 + 2xv'\cos x^2$

$$\cancel{2vx\cos x^2} + 2x^2v'\cos x^2 - \cancel{4x^3v\sin x^2} + xv''\sin x^2 + 2x^2v'\cos x^2 - \cancel{2xv\cos x^2}$$
$$- v'\sin x^2 + \cancel{4x^3v\sin x^2} = 0$$

Ex. (27) $xy'' - y' + 4x^3y = 0,\ t > 0$ $\qquad y_1(x) = \sin(x^2)$

Put $y = v(x)y_1(x) = v(x)\sin(x^2)$

$$y' = v'(x)\sin(x^2) + 2x\,v(x)\cos(x^2)$$

$$y'' = v''(x)\sin(x^2) + 2x\,v'(x)\cos(x^2) + (2x\,v(x))'\cos(x^2) + 4x^2v(x)(-\sin(x^2))$$

$$= v''(x)\sin(x^2) + 2x\,v'(x)\cos(x^2) + 2v(x)\cos(x^2) + 2x\,v'(x)\cos(x^2) - 4x^2v(x)\sin(x^2)$$

$$= v''(x)\sin(x^2) + v'(x)\left[4x\cos(x^2)\right] + v(x)\left[2\cos(x^2) - 4x^2\sin(x^2)\right]$$

The DE becomes:

$$x\,v''(x)\sin(x^2) + v'(x)\left[4x^2\cos(x^2) - \sin(x^2)\right] + v(x)\left[\cancel{2x\cos(x^2)} - \cancel{4x^3\sin(x^2)} - \cancel{2x\cos(x^2)}\right]$$
$$+ \cancel{4x^3 v(x)\sin(x^2)} = 0$$

or

$$\frac{dv'}{dx}\left(x\sin(x^2)\right) + v'\left(4x^2\cos(x^2) - \sin(x^2)\right) = 0$$

$$\frac{dv'}{dx} + v'\left[\frac{4x^{\cancel{2}}\cos(x^2)}{\cancel{x}\sin(x^2)} - \frac{\cancel{\sin(x^2)}}{x\cancel{\sin(x^2)}}\right] = 0$$

$$\frac{dv'}{dx} + v'\left(4x\cot(x^2) - \frac{1}{x}\right) = 0$$

$$\frac{dv'}{v'} = \Big(1/x - \underbrace{4x}_{2du}\cot(\underbrace{x^2}_{u})\Big)\,dx$$

$$\ln(v') = \ln x - 2\ln(\sin(x^2)) + C$$

$$\ln(v') = \ln\left(\frac{x}{\sin^2(x^2)}\right) + C$$

or $v' = \dfrac{Cx}{\sin^2(x^2)}$

Thus $v = \frac{1}{2}\displaystyle\int \frac{\overset{du}{2x}}{\sin^2(\underbrace{x^2}_{u})}\,dx = \frac{1}{2}\int \csc^2(u)\,du = -\frac{1}{2}\cot u$

$$= -\frac{1}{2}\cot(x^2)$$

Thus, $y = vy_1 = -\frac{1}{2}\cot(x^2)\sin(x^2) = -\frac{1}{2}\cos(x^2)$

but the $(-\frac{1}{2})$ doesn't really matter

The general sol'n of the DE is:

$y = C_1\sin(x^2) + C_2\cos(x^2)$.

Ex (28) $(x-1)y'' - xy' + y = 0 \qquad y_1(x) = e^x$

Guess: $y = v e^x$

$y' = v' e^x + v e^x$

$y'' = v'' e^x + 2v' e^x + v e^x$

Plugging back in:

$(x-1)(v'' e^x + 2v' e^x + v e^x) - x(v' e^x + v e^x) + v e^x = 0$

$x v'' e^x + 2x v' e^x + \cancel{x v e^x} - v'' e^x - 2v' e^x - \cancel{v e^x} - x v' e^x - \cancel{x v e^x} + \cancel{v e^x} = 0$

$(x-1) v'' e^x + (x-2) v' e^x = 0$

$(x-1) v'' + (x-2) v' = 0$

$(x-1)\dfrac{dv'}{dx} = (2-x) v'$

$\dfrac{dv'}{v'} = \dfrac{2-x}{x-1}\, dx$

$\int \dfrac{-x}{x-1}\, dx \qquad u = x-1, \quad x = u+1, \quad dx = du$

$\int \dfrac{-u-1}{u}\, du = \int -1 - \frac{1}{u}\, du$

$= -u - \ln u$

$= -x+1 - \ln(x-1)$

$\ln v' = 2\ln(x-1) - x + 1 - \ln(x-1) + C$

$= \ln(x-1) - x + 1 + C$

$\Rightarrow v' = C(x-1)e^{1-x}$

$v' = C(1-x)e^{1-x}$

$v = C\int (1-x)e^{1-x}\, dx \qquad u = 1-x, \quad du = -dx$

$C_1\int u e^u\, du = C_1(ue^u - e^u) + C_2$

$= C_1\left[(1-x)e^{1-x} - e^{1-x}\right] + C_2$

so $v = C_1\left[(1-x)e^{1-x} - e^{1-x}\right] + C_2$

and $y = v e^x = C_1\left[(1-x)e - e\right] + C_2 e^x$

$= \cancel{C_1 e} - C_1 x - \cancel{C_1 e} + C_2 e^x = C_1 x + C_2 e^x$

Thus $y_2(x) = x$.

Check $W(y_1, y_2) = W(x, e^x) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = x e^x - e^x = e^x(x-1)$

$\neq 0$ if $x \neq 1$.

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