3.3. Characteristic Eq'ns w/ complex roots

Suppose the characteristic eq'n $ar^2 + br + c = 0$ has roots $\lambda \pm i\mu$.

Then the solution should be $y = e^{(\lambda \pm i\mu)t}$

Let's examine this. We can write $y = e^{\lambda t} e^{\pm i\mu t}$

First, look at e^{it}

The power series expansion of $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

So the expansion of $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!}$

writing out some terms:

$$e^{it} = 1 + it + \frac{(-1)t^2}{2} + \frac{-i\,t^3}{2\cdot 3} + \frac{(1)t^4}{2\cdot 3\cdot 4} + \ldots$$

Separating out the i's from the non-i's:

$$e^{it} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}}_{\cos t} + i \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}}_{\sin t}$$

These are the power series for cos t and sin t!

Thus, $e^{it} = \cos t + i\sin t$ for all $t \in \mathbb{R}$!

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$$

and $e^{\pm i\mu t} = \cos(\mu t) \pm i\sin(\mu t)$ ← [This is Euler's Formula. What to blow your mind? Calculate $e^{i\pi}$.]

Going back to our solution

$$y = e^{\lambda} e^{\pm i\mu t} = e^{\lambda}\left(\cos(\mu t) \pm i\sin(\mu t)\right)$$

But this is missing the unknown constants. We should get one for the + and one for the − signs. It turns out you can use some algebraic maneuvering to write the general solution as:

$$\begin{cases} y_1 = e^{\lambda}\left(C_1\cos(\mu t) + iC_2\sin(\mu t)\right) \\ y_2 = e^{\lambda}\left(C_1\cos(\mu t) - iC_2\sin(\mu t)\right) \end{cases}$$

or just $\boxed{y = e^{\lambda}\left(C_1\cos(\mu t) \pm iC_2\sin(\mu t)\right)}$

Examples:

7. $y'' - 2y' + 2y = 0$
12. $4y'' + 9y = 0$
14. $9y'' + 9y' - 4y = 0$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Fixing yesterday's mess:

We found the solution:

$$y = e^{\lambda t}\left(C_1\cos(\mu t) \pm i\, C_2 \sin(\mu t)\right)$$

where $\lambda \pm i\mu$ were complex roots of the characteristic equation.

In general, a solution of the SODE

$$ay'' + by' + cy = 0$$

is an integral curve in the (t,y) plane, hence a <u>real-valued expression</u>.

Therefore we don't want complex i's in our solution. We can require that C_2 be purely imaginary, or replace iC_2 by a "new" C_2 to get the general sol'n:

$$\boxed{y = e^{\lambda t}\left(C_1 \cos(\mu t) \mp C_2 \sin(\mu t)\right)}$$

let C_2 absorb the $\pm$.

<u>More Ex.</u>

19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2.$