3.2. Sol'ns of Linear Homogeneous Equations; The Wronskian

Let p and q be continuous functions on an interval $\alpha<t<\beta$. Then for any function φ that is twice differentiable on I, we define a differential operator $\mathcal{L}$ by

$$\mathcal{L}\varphi = \varphi'' + p\varphi' + q\varphi$$

Notice: $\mathcal{L}\varphi$ is a function on $\alpha<t<\beta$.

Ex. $p(t)=t^2$, $q(t)=1+t$, $\varphi(t)=\sin 3t$

$$\begin{aligned}\mathcal{L}\varphi &= \varphi'' + t^2\varphi' + (1+t)\varphi \\ &= (\sin 3t)'' + t^2(\sin 3t)' + (1+t)(\sin 3t) \\ &= -9\sin 3t + 3t^2\cos 3t + \sin 3t + t\sin 3t\end{aligned}$$

We often write $\mathcal{L}$ in the following way:

$$\mathcal{L} = \partial^2 + p\,\partial + q$$

Now we can write a homogeneous SODE in the form

$$\mathcal{L}y = y'' + p(t)y' + q(t)y = 0$$

and and IVP:

$$y(t_0)=y_0$$
$$y'(t_0)=y_0'$$

Thm. (FEUT) Consider the IVP

$$\mathcal{L}y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \quad \text{given}$$

where p, q, g are continuous functions on an interval I that contains t_0. Then there is exactly one sol'n $y = \varphi(t)$ of this problem, and it exists on all of I.

If we can actually find the sol'n, then this is easy to verify.

Ex. $y'' - y = 0$, $y(0) = 2$, $y'(0) = -1$. Solve it.

But this thm can be useful in other ways.

Ex. $(t^2 - 3t)y'' + ty' - (t+3)y = 0$, $y(1) = 2$, $y'(1) = 1$

Find the longest interval on which the sol'n is certain to exist.

$(0, 3)$

Ex. Find the unique sol'n of

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0.$$

$\varphi(t) = 0$ for all $t \in I$ works. By the FEUT it is the only sol'n.

Thm. (Superposition) If y_1 and y_2 are two sol'ns of

$$\mathcal{L}y = 0,$$

then the linear combination $c_1 y_1 + c_2 y_2 = y$ is also a sol'n.

Prove the thm by computing $\mathcal{L}(c_1y_1+c_2y_2)$.

Using the sol'n $y=c_1y_1+c_2y_2$ we can write an IVP:

$$\mathcal{L}y=0$$
$$\begin{cases} c_1y_1(t_0)+c_2y_2(t_0)=y_0 \\ c_1y_1'(t_0)+c_2y_2'(t_0)=y_0' \end{cases}$$

We want to know if we can find c_1 and c_2 that satisfy this. Thinking of c_1 and c_2 as the variables, we can find the determinant of the system:

$$W=\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0)-y_2(t_0)y_1'(t_0)$$

The system has a unique sol'n (c_1,c_2) if $W\neq 0$, regardless of the values of y_0 and y_0'. W is called a <u>Wronskian</u>.

The sol'ns are given by <u>Cramer's Rule</u> by:

$$c_1=\frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W},\quad c_2=\frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}.$$

<u>Ex.</u> Verify that $y_1(t)=t^2$ and $y_2(t)=t^{-1}$ are two sol'ns of DE $t^2y''-2y=0$, $t>0$. Then show that $y=c_1t^2+c_2t^{-1}$ is also a sol'n for any c_1 and c_2.

The Cramer's Rule argument establishes the following thm.—

Thm. Let y_1 and y_2 be two sol'ns of

$$\left.\begin{array}{l} \mathcal{L}[y] = y'' + p(t)\,y' + q(t)\,y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0'. \end{array}\right\} (*)$$

Then it is always possible to choose the constants c_1, c_2 such that

$$y = c_1 y_1 + c_2 y_2$$

solves $(*)$ iff the Wronskian

$$W = y_1 y_2' - y_2 y_1' \neq 0$$

at the initial point t_0.

Ex. $\begin{cases} y'' + 5y' + 6y = 0 \\ y_1 = e^{-2t} \\ y_2 = e^{-3t} \end{cases}$ $\quad W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -3e^{-5t} + 2e^{-5t} = -e^{-5t}$

$W \neq 0$ for any value of t, thus $y = c_1 e^{-2t} + c_2 e^{-3t}$ is a general sol'n for all t.

Thm. Suppose y_1 and y_2 are two sol'ns of

$$\mathcal{L}[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

Then the family of solutions

$$y = c_1 y_1 + c_2 y_2$$

includes <u>every</u> solution iff $\exists$ a point t_0 where $W(y_1, y_2)(t_0) \neq 0$.

We'll believe this without proof. Or will we?

Let φ be a sol'n. We must show that there are constants C_1 and C_2 such that $\varphi = C_1 y_1 + C_2 y_2$.

Let t_0 be the point where the Wronskian $\neq 0$. Put

$$y_0 = \varphi(t_0), \quad y_0' = \varphi'(t_0).$$

Next consider the IVP:

$$\mathcal{L}[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

Obviously φ solves this SODE. By the previous thm, we can choose C_1, C_2 such that $y = C_1 y_1 + C_2 y_2$ is also a sol'n.

Then the FEUT says that the two sol'ns φ and $C_1 y_1 + C_2 y_2$ must be equal (uniqueness). ∎

We call $y = C_1 y_1 + C_2 y_2$ the general sol'n of the SODE, and the solutions $\{y_1, y_2\}$ are called the fundamental solution set.

Ex. (26) $x^2 y'' - x(x+2) y' + (x+2) y = 0$

$$y_1 = x$$
$$y_2 = x e^x$$

Verify that the functions solve the SODE. Do they form a fundamental sol'n set?

Ex. $ay'' + by' + cy = 0$

$y_1 = e^{r_1 t} \quad y_2 = e^{r_2 t}, \quad r_1 \neq r_2.$

Show that these form fund. sol'n. set.

Ex. $y_1 = t^{1/2}, \; y_2 = \frac{1}{t}, \quad 2t^2 y'' + 3ty' - y = 0, \quad t > 0.$

Thm. Consider the SODE: $\mathcal{L}[y] = 0$. p, q continuous on I, $t_0 \in I$.

Let y_1 solve the IVP w/ $y(t_0) = 1, \; y'(t_0) = 0$
and y_2 solve the IVP w/ $y(t_0) = 0, \; y'(t_0) = 1$
Then y_1 and y_2 form a fund. sol'n set for $\mathcal{L}[y] = 0$.

Proof. Calculate the Wronskian. ∎

Ex. $y'' - y = 0, \quad t_0 = 0.$

We already know that $y_1 = e^t$ and $y_2 = e^{-t}$ are sol'ns.
Their Wronskian $= -2 \neq 0$. So they form a fundamental sol'n set.

The sol'n to the IVP w/ $y(0) = 1, y'(0) = 0$ is
$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t)$.

The sol'n to the IVP w/ $y(0) = 0, y'(0) = 1$ is
$y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t$

$W(y_3, y_4)(t) = \sinh^2 t - \cosh^2 t = 1$

Thus, the general sol'n can be written $\boxed{y = k_1 \cosh t + k_2 \sinh t}$

Does this contradict the FEUT?

Next: Abel's Theorem. If y_1 and y_2 are sol'ns of the SODE

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p, q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = C \exp\left[-\int p(t)\,dt\right] = C e^{-\int p(t)dt} \qquad (*)$$

where C is a constant that depends on the functions y_1 and y_2, but not t.

Furthermore, $W(y_1, y_2)(t)$ is either always 0 or never 0 on I.

Proof: Next project! ∎

Ex. $2t^2 y'' + 3ty' - y = 0 \quad , \quad t>0$

$y_1 = t^{1/2} \quad , \quad y_2 = 1/t$

We found earlier that the Wronskian of these two is given by: $-\frac{3}{2} t^{-3/2}$

Check formula $(*)$ for this sol'n set.

$$W(y_1, y_2)(t) = e^{-\int \frac{3}{2}\frac{1}{t}\,dt} = e^{-\frac{3}{2}\ln t + c} = c\, e^{\ln t^{-3/2}} = c\, t^{-3/2}.$$

This gives the Wronskian for any pair of fund. sol'ns.

Our pair has $C = -3/2$.

Notice, if $C \neq 0$, then W is defined on all of $(0, \infty)$.

Ex. (20) If the Wronskian of f and g is $t\cos t - \sin t$, Find the Wronskian of $u = f + 3g$ and $v = f - g$.

$$W(u,v) = \begin{vmatrix} f+3g & f-g \\ f'+3g' & f'-g' \end{vmatrix} = (f+3g)(f'-g') - (f-g)(f'+3g')$$

$$= \cancel{ff'} - fg' + 3gf' - \cancel{3gg'} - \cancel{ff'} - 3fg' + gf' + \cancel{3gg'}$$

$$= 3(gf' - fg') + (gf' - fg')$$

$$= 4(gf' - fg') = -4W(f,g)$$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

$$= \boxed{4\sin t - 4t\cos t}$$

Ex. 31. <u>Bessel's Equation</u> $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

Find the Wronskian w/out solving the SODE.

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$$W = e^{-\int \frac{1}{x}dx} = \frac{C}{x}$$

32. <u>Legendre's Eqn.</u> $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

$$W = e^{\int \frac{2x}{1-x^2}dx} = e^{-\ln(1-x^2)+C} = \frac{C}{1-x^2}$$