# Chapter 3: Second Order Linear Equations

## 3.1: Homogeneous Equations w/ Constant Coefficients

A second order ODE (SODE) has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad \text{or} \qquad (*)$$
$$y'' = f(t, y, y')$$

A <u>linear</u> $2^{nd}$ order ODE has the form

$$y'' = g(t) - p(t)\,y' - q(t)\,y \qquad \text{or}$$

$$R(t)\,y'' + P(t)\,y' + Q(t)\,y = G(t) \qquad (**)$$

$$\text{where } p(t) = \frac{P(t)}{R(t)}, \quad q(t) = \frac{Q(t)}{R(t)}, \quad g(t) = \frac{G(t)}{R(t)}$$

so this can be written as

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

An <u>initial value problem</u> (IVP) now requires <u>two</u> initial conditions (since there are two derivatives being "undone"). One form of initial data looks like:

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

Initial conditions on both $y$ and $y'$ at $t_0$.

A SODE is called <u>homogeneous</u> if $G(t)=0$ in (**).
A homog. SODE then looks like:

$$P(t)\,y'' + Q(t)\,y' + P(t)\,y = 0.$$ (changed letters to match the book)

If $P, Q, R$ are all constants, then this SODE becomes

$$ay'' + by' + cy = 0. \qquad (***)$$

An RE from Chapter 2:

<u>Ex.</u> Solve the SODE $y''-y=0$ and the IVP: $\begin{cases} y''-y=0 \\ y(0)=2 \\ y'(0)=-1 \end{cases}$

Rewrite as $y''=y$

The function whose 2nd derivative equals itself are:

$$y_1 = C_1 e^{t}$$
$$y_2 = C_2 e^{-t}$$

<u>RE.</u> Verify this!

The general sol'n is thus: $y = C_1 e^{t} + C_2 e^{-t}$, $y' = C_1 e^{t} - C_2 e^{-t}$

Now plug in the ICs: $\begin{cases} 2 = C_1 + C_2 \\ -1 = C_1 - C_2 \end{cases}$ solve the system to find $C_1, C_2$.

$$1 = 2C_1$$
$$C_1 = \tfrac{1}{2} \Rightarrow C_2 = \tfrac{3}{2}$$

And the sol'n of the IVP is:

$$\boxed{y = \tfrac{1}{2}e^{t} + \tfrac{3}{2}e^{-t}}$$

Now, back to the general eqn (***):

$$ay'' + by' + cy = 0$$

We guess that $y = e^{rt}$ will be a sol'n for some $r$.
Then $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. Plug into the SODE to get:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$(ar^2 + br + c)\, e^{rt} = 0$$

This can only $= 0$ if $\boxed{ar^2 + br + c = 0}$.
This is called the <u>Characteristic Equation</u> of the SODE.

This is just a quadratic equation. It has two solutions (in general) $r_1$ and $r_2$. These are the values of $r$ that make $y = e^{rt}$ a sol'n of (***). Thus, the general sol'n of (***) is:

$$\boxed{y = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

Our job is just to find $r_1, r_2, C_1, C_2$.

<u>Ex.</u> $y'' + 5y' + 6y = 0$

Characteristic eq'n is: $r^2 + 5r + 6 = 0$

$$(r+2)(r+3) = 0$$
$$r = -2, -3$$

sol'n is: $\boxed{y = C_1 e^{-2t} + C_2 e^{-3t}}$

Ex. Solve the IVP: $\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$

We found $y = C_1 e^{-2t} + C_2 e^{-3t}$ as our sol'n

$y' = -2C_1 e^{-2t} - 3C_2 e^{-3t}$

$y(0) = 2 \Rightarrow 2(C_1 + C_2 = 2)$

$y'(0) = 3 \Rightarrow -2C_1 - 3C_2 = 3$

$-C_2 = 7$

$C_1 = +9$

Sol'n: $\boxed{y = +9e^{-2t} - 7e^{-3t}}$

Ex. $\begin{cases} 4y'' - 8y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = 1/2 \end{cases}$

Characteristic Eqn:

$4r^2 - 8r + 3 = 0$

$r = \dfrac{8 \pm \sqrt{64 - 48}}{8} = \dfrac{8 \pm \sqrt{16}}{8} = \dfrac{12}{8}, \dfrac{4}{8} = \dfrac{3}{2}, \dfrac{1}{2}$

$y = C_1 e^{1/2 t} + C_2 e^{3/2 t}$

$y' = \frac{1}{2} C_1 e^{1/2 t} + \frac{3}{2} C_2 e^{3/2 t}$

ICs $\Rightarrow \begin{cases} C_1 + C_2 = 2 \\ \frac{1}{2} C_1 + \frac{3}{2} C_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 2 \\ C_1 + 3C_2 = 1 \end{cases}$

$2C_2 = -1 \Rightarrow C_2 = -1/2$

$C_1 = 5/2$

Sol'n is: $\boxed{y = \frac{5}{2} e^{1/2 t} - \frac{1}{2} e^{3/2 t}}$

Notes: • When $r_1 = r_2$, this method doesn't quite work. We'll deal with that soon.

• when $r_1, r_2 \in \mathbb{C}$ (are complex) we need to use some trig/complex identities to get the solution. Again, we'll do this soon. ☺

Exs.

(17) Find a SODE w/ sol'n $y = c_1 e^{2t} + c_2 e^{-3t}$

Any of 1-16.