2.8: The Existence and Uniqueness Theorem

This section discusses the proof of the FEUT (from section 2.4) for general first order initial value problems.

The theorem says that, under certain conditions, the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution in some open interval containing t_0.

In case $y' = f(t,y)$ is a linear DE, then we already discussed a sketch of the proof: It follows from the continuity of all of the functions in the general solution,

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds + \frac{C}{\mu(t)}.$$

If $y' = f(t,y)$ is nonlinear, then we need a different method.

First, we may always assume that the initial condition is

$$y(0) = 0.$$

If not, we can make a change of variables that corresponds to a translation of the plane geometrically.

Ex. $y' = 1 - y^3$, $y(-1) = 3$

Transform the IVP to an equivalent problem w/ initial point at the origin.

$t \mapsto t_2 + 1$ $\quad dt = dt_2$

$y \mapsto y_2 - 3$ $\quad \frac{dy}{dt} = \frac{dy_2}{dt_2}$

$\leftarrow \Big\{$ bad notation! sorry :(

so the IVP becomes

$$y_2' = 1 - (y_2 - 3)^3$$

$$y_2' = 1 - (y_2^3 - 3y_2^2 + 9y_2 - 27)$$

$$y_2' = -y_2^3 + 3y_2^2 - 9y_2 + 28$$

Sub $y = y_2$ to get the IVP:

$$\begin{cases} y' = -y^3 + 3y^2 - 9y + 28 \quad = 1 - (y-3)^2 \\ y(0) = 0 \end{cases}$$

Now that we can assume the initial value of our DE is $y(0) = 0$, then we can rewrite the FEUT as follows:

Thm. (FEUT). If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R: |t| \le a$, $|y| \le b$ around the origin, then there is a unique solution $y = \varphi(t)$ of the IVP:

$$\begin{cases} y' = f(t, y) \\ y(0) = 0 \end{cases}$$

Next, we temporarily make another assumption to make the thm. easier to prove:

Suppose that there is a <u>differentiable</u> function $y = \varphi(t)$ that satisfies the IVP; then $f(t,y) = f(t,\varphi(t))$ is a continuous function of t only. Hence we can integrate $y' = f(t,y)$ from $t=0$ to obtain:

$$\varphi(t) = \int_0^t f[s,\varphi(s)]\, ds \qquad (**)$$

by the FTC. Here we use that $\varphi(0)=0$ (so there is no $+ y_0$).

Eqn $(**)$ is called an <u>integral equation</u> since it relates an unknown function φ to its own integral.

The integral $(**)$ is <u>not</u> a solution of the IVP, but it does provide another relation satisfied by any solution of the IVP.
Conversely, if there is a continuous function $y=\varphi(t)$ that solves the IVP, then it satisfies eq $(**)$.

First, notice that $\varphi(0) = \int_0^0 f[s,\varphi(s)]\, ds = 0$, so the initial condition is satisfied.
Second, by the FTC, since φ is continuous,

$$\frac{d\varphi}{dt} = \frac{d}{dt}\int_0^t f(s,\varphi(s))\, ds = f(t,\varphi(t))$$

which holds since φ is a solution of $(*)$.
Thus, $(*)$ and $(**)$ are equivalent in the sense that a solution of 1 solves the other.

It is ~~usually~~ more convient to show that the solution integral eqn (**) is unique on some interval $t \leq |h|$.

The method for showing uniqueness is called the method of successive approximations, or Picard's iteration method.

Step 1. Choose an arbitrary function, usually

$$\varphi_0(t) = 0.$$

Then φ_0 satisfies the initial condition in (*), although ~~prob~~ Probably not the DE itself.

Step 2. Obtain a new approximation $\varphi_1(t)$ by evaluating:

$$\varphi_1(t) := \int_0^t f[s, \varphi_0(s)]\, ds$$

Step 3. Continue finding new approximations $\varphi_2, \varphi_3, \ldots$ by defining

$$\varphi_n(t) = \int_0^t f[s, \varphi_{n-1}(s)]\, ds.$$

We'll obtain a sequence of continuous functions $\{\varphi_n\} = \varphi_0, \varphi_1, \varphi_2, \ldots$ that all satisfy the initial condition $\varphi_n(0) = 0$. In general, none of these will be an actual solution to the DE (*).

However!, if at any stage $n=k$ we find that $\varphi_k = \varphi_{k+1}$, then this is a solution of (**), hence also (*).

In general, we have to consider the infinite sequence $\{\varphi_n\}$, the limit of which should be our desired solution. Unfortunately, the limit may not be continuous, cf. RE # 13.

We won't ~~should~~ show that the sequence converges in class, but ~~we~~ ~~will~~ the process is outlined in REs 15-18.

Ex (8). Determine $\varphi_n(t)$ for arbitrary n.

$$y' = t^2 y - t \quad , \quad y(0) = 0$$

$$\varphi_0(t) = 0$$

$$\varphi_1(t) = \int_0^t -s \, ds = -\tfrac{1}{2} s^2 \Big|_0^t = -\tfrac{1}{2} t^2$$

$$\varphi_2(t) = \int_0^t -\tfrac{1}{2} s^4 - s \, ds = -\tfrac{1}{10} t^5 - \tfrac{1}{2} t^2$$

$$\varphi_3(t) = \int_0^t -\tfrac{1}{10} s^7 - \tfrac{1}{2} s^4 - s \, ds = -\tfrac{1}{80} t^8 - \tfrac{1}{10} t^5 - \tfrac{1}{2} t^2$$

$$\varphi_4(t) = \varphi_3(t) + \left(\frac{-1}{2 \cdot 5 \cdot 8 \cdot 11} t^{11} \right)$$

so $$\varphi_n(t) = \varphi_{n-1}(t) - \frac{1}{2 \cdot 5 \cdots (3n-1)} t^{(3n-1)}$$

or

$$\boxed{\varphi_n(t) = \sum_{i=1}^{n} \frac{-1}{2 \cdot 5 \cdots (3i-1)} t^{3i-1}}$$

Ex. (9) Calculate $\varphi_1, \varphi_2, \varphi_3$ for $\varphi_0 = 0$:

$$y' = t^2 + y^2, \quad y(0) = 0.$$

$$\varphi_0(t) = 0$$
$$\varphi_1(t) = \int_0^t s^2\, ds = \tfrac{1}{3}t^3$$
$$\varphi_2(t) = \int_0^t s^2 + \tfrac{1}{3}s^3\, ds = \tfrac{1}{3}t^3 + \frac{1}{3\cdot 4}t^4$$
$$\varphi_3(t) = \int_0^t s^2 + \tfrac{1}{3}s^3 + \frac{1}{3\cdot 4}s^4\, ds = \tfrac{1}{3}t^3 + \frac{1}{3\cdot 4}t^4 + \frac{1}{3\cdot 4\cdot 5}t^5$$

Do the iterations appear to be converging? to what?

Recall that $e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!} = 1 + t + \frac{1}{2}t^2 + \frac{1}{2\cdot 3}t^3 + \frac{1}{2\cdot 3\cdot 4}t^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5}t^5 + \ldots$

thus $2e^t = 2 + 2t + t^2 + \underbrace{\frac{1}{3}t^3 + \frac{1}{3\cdot 4}t^4 + \frac{1}{3\cdot 4\cdot 5}t^5 + \ldots}_{\text{This is what we have for } \varphi_n(t).}$

Thus the sol'n $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$ should be

$$\varphi(t) = 2e^t - 2 - 2t - t^2$$
$$\varphi'(t) = 2e^t - 2 - 2t$$
$$(\varphi(t))^2 = \begin{array}{r} 2e^t - 2 - 2t - t^2 \\ 2e^t - 2 - 2t - t^2 \\ \hline -2t^2e^t + 2t^2 + 2t^3 + t^4 \\ -4te^t + 4t \quad + 4t^2 + 2t^3 \\ -4e^t + 4 \quad + 4t \quad + 2t^2 \\ +4e^{2t} \quad -4e^t \quad -4te^t \quad -2t^2e^t \\ \hline \end{array}$$

hmm. Something wrong here :(