2.7: Numerical Approximations: Euler's Method

Consider the first order IVP:

$$\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases} \qquad (*)$$

Recall two basic facts:

1. If f and $\partial f/\partial y$ are continuous at (t_0, y_0), then the IVP $(*)$ has a unique solution in some interval about t_0.

2. It is usually not possible to write down a solution φ of the DE, even when we know it exists.

However, it is always possible for first order problems to draw a direction field and look for integral curves. These curves are the real (actual) solutions to the DE that are reprepresented by expressions $\varphi(x,y)$ when one can be found. Thus, if we can find the integral curve that solves the IVP $(*)$, then we don't care if we have an expression φ.

Ex. $\frac{dy}{dt} = 3 - 2t - \frac{1}{2}y$

Look at the slope field in the book.

Try to visualize some solutions.

To approximate solutions to a DE, we can use Euler's method: We form a piecewise linear curve made out of the tangent lines in the slope field.

Recall from Calc I:

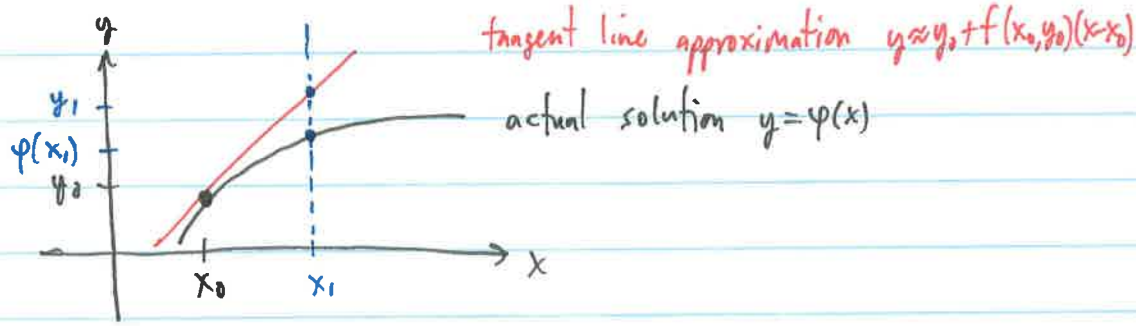
The tangent line approximation to a curve y at the point (x_0, y_0) is given by

$$y \approx y_0 + \left(\frac{dy}{dx}(x_0,y_0)\right)(x-x_0)$$

but the derivative $y' = \frac{dy}{dx}$ is given by $f(x,y)$ in the DE, so we have

$$y \approx y_0 + f(x_0,y_0)\cdot(x-x_0)$$

The situation looks like this



As x_1 gets further away from x_0, it is possible that y_1 will get very far away from $\varphi(x_1)$. But locally (close to x_0), $y_1 \approx \varphi(x_1)$.

We want to build a curve in this fashion, by moving "a little ways" along a tangent line, then stopping and picking up along a new tangent line: the one prescribed by the DE at that point.

Between two x-values x_1 and x_2, we can write:

$$y \approx y_1 + f(x_1, y_1)(x - x_1)$$

Then the value y_2 would be

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$

We can then use (x_2, y_2) to make a new tangent line:

$$y = y_2 + f(x_2, y_2)(x - x_2)$$

and move along x until we get to x_3. The value y_3 is then

$$y_3 = y_2 + f(x_2, y_2)(x_3 - x_2)$$

And so on. We can keep repeating this process until we get to a desired x-value $b = x_N$. The piecewise linear curve that results will be an <u>approximate</u> solution of the DE.

The more "partition points" we choose, the closer our approximation will be to an actual solution.

We can write a recursively defined formula for y_n:

$$y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n)$$

Setting $f_n := f(x_n, y_n)$ and taking a uniform partition with $(x_{n+1} - x_n) = h$

we can write:

$$y_{n+1} = y_n + f_n \cdot h \qquad n = 0, 1, 2, \ldots$$

To use Euler's method, choose a partition of your desired domain $[a,b]$, then calculate the y-values using the preceding method. Plot all points (x_i, y_i), then connect them all with straight lines.

Ex. $\begin{cases} y' = 3 - 2t - \frac{1}{2}y \\ y(0) = 1 \end{cases}$ Use $h = 0.2$ to approximate the sol'n on $[0,1]$.

(*) Use Excel to do this in class.

Ex. Change the step size to $h = 0.1$, $h = 0.05$, $h = 0.025$, $h = 0.01$. Plot all approximations on the same graph (if possible).

Ex. $\frac{dy}{dt} = 4 - t + 2y \qquad y(0) = 1.$