We're skipping 2.5, so

2.6: Exact Equations and Integrating Factors

We already know two methods for solving "special" first order ODE: integrating factors and separation of variables.
If an equation is not separable, but exact, then there is another method.

Ex. Solve the DE: $2x + y^2 + 2xyy' = 0$

This equation is neither separable nor linear.

Consider the function $\psi(x,y) = x^2 + xy^2$
This has partial derivatives:

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \quad \text{and}$$
$$\frac{\partial \psi}{\partial y} = 2xy$$

So the DE can be rewritten as

$$\cancel{\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}}$$

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \qquad (*)$$

Not necessary! $\Big\{$ Multiplying through by dx we have $\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$.

Assuming that $y = y(x)$ and using the chain rule, we can rewrite this as

$$\frac{d\psi}{dx} = \frac{d}{dx}\left(x^2 + xy^2\right) = 0$$

Therefore $\psi(x,y) = \boxed{x^2 + xy^2 = c}$

is an implicitly defined solution for the DE.

The key step is recognizing that there is a function ψ that satisfies eq'n $(*)$.

In general, if we start w/ an eq'n of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (\text{from } \S 2.2) \qquad (**)$$

then $(**)$ is called <u>exact</u> if there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

The sol'n is given by $\psi(x,y) = c$, where c is an arbitrary constant.

<u>Question</u>: How do we find $\psi(x,y)$!?
If such a ψ even exists.

Thm. Let $M, N, \partial_y M$, and $\partial_x N$ be continuous on a rectangular region $R: (\alpha, \beta) \times (\gamma, \delta)$. Then the DE

$$M(x,y) + N(x,y)\, y' = 0$$
on R

is exact, if and only if $\partial_y M = \partial_x N$ for each pt in R.

Ex. Solve the DE: $\underbrace{(y\cos x + 2xe^y)}_{M(x,y)} + \underbrace{(\sin x + x^2 e^y - 1)}_{N(x,y)} y' = 0.$

$$\left.\begin{aligned} \partial_y M &= \cos x + 2xe^y \\ \partial_x N &= \cos x + 2xe^y \end{aligned}\right\} \partial_y M = \partial_x N \Rightarrow \text{DE is exact.}$$

Now we need to find the sol'n $\psi(x,y)$. To do so, Calculate $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.
These should be equal once appropriate constants are chosen.

$$\int M(x,y)\,dx = \int y\cos x + 2xe^y\,dx = y\sin x + x^2 e^y + h_1(y)$$

$$\int N(x,y)\,dy = \int \sin x + x^2 e^y - 1\,dy = y\sin x + x^2 e^y - y + h_2(x)$$

For these to be equal $h_1(y) = -y$ and $h_2(x) = 0$.

Thus $\psi(x,y) = y\sin x + x^2 e^y - y$ and the sol'n

is $\boxed{y\sin x + x^2 e^y - y = C}$

$\underline{\text{Ex.}}$ $(3xy + y^2) + (x^2 + xy)\, y' = 0$

$$\left.\begin{array}{l} \partial_y M = 3x + 2y \\ \partial_x N = 2x + y \end{array}\right\} \quad \begin{array}{l} \partial_y M \neq \partial_x N \quad \text{so this is } \underline{\text{not}} \text{ exact,} \\ \text{and cannot be solved by this method.} \end{array}$$

We may be able to convert it to an exact eq'n by using an integrating factor (cf. §2.1).

We want to find a function $\mu(x,y)$ such that

$$\mu(x,y)\, M(x,y) + \mu(x,y)\, N(x,y)\, y' = 0 \qquad (**)$$

is an exact DE. Then what we need is:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\text{or} \quad M \mu_y - N \mu_x + (M_y - N_x)\mu = 0$$

If a μ satisfying this DE can be found then $(**)$ is exact, and we can solve it.

If we assume that $\mu = \mu(x)$ is not a function of y, then

$$\frac{\partial}{\partial y}(\mu M) = \mu M_y \quad \text{and} \quad \frac{\partial}{\partial x}(\mu N) = \mu N_x + N \frac{d\mu}{dx}$$

Thus, for $\frac{\partial}{\partial y}(\mu M)$ to equal $\frac{\partial}{\partial x}(\mu N)$ we need

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \qquad (***)$$

If $\frac{M_y - N_x}{N}$ is a function of only x, then $\exists$ a $\mu(x)$ and $(***)$ is separable, so such a μ can be found.

A similar procedure can be used to find a $\mu = \mu(y)$.

Back to the last problem:

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

$$\partial_y M = M_y = 3x + 2y$$
$$\partial_x N = N_x = 2x + y$$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x+y)} = \frac{1}{x}$$

Thus, there is a $\mu = \mu(x)$ that satisfies the DE

$$\mu' = \frac{\mu}{x}, \quad \text{or} \quad \frac{d\mu}{\mu} = \frac{dx}{x}$$

and $\mu = x$.

Multiplying through: $(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$

Now: $$\left.\begin{aligned} \frac{\partial}{\partial y}(\mu M) &= 3x^2 + 2xy \\ \frac{\partial}{\partial x}(\mu N) &= 3x^2 + 2xy \end{aligned}\right\} \; !$$

Then ψ is given by:

$$\int \mu M \, dx = \int 3x^2 y + xy^2 \, dx = x^3 y + \tfrac{1}{2} x^2 y^2 + h_1(y)$$

$$\int \mu N \, dy = \int x^3 \cancel{y} + x^2 y \, dy = x^3 y + \tfrac{1}{2} x^2 y^2 + h_2(x)$$

So the sol'n is given by:

$$\psi(x,y) = x^3 y + \tfrac{1}{2} x^2 y^2$$

$$\Rightarrow \quad \boxed{x^3 y + \tfrac{1}{2} x^2 y^2 = C}$$

RE. Verify that this sol'n solves the <u>original</u> DE

$$(3xy + y^2) + (x^2 + xy)\, y' = 0$$

by differentiating the sol'n implicitly. You should get:

$$y' = \frac{-(3xy + y^2)}{x^2 + xy}$$