2.4: Differences between Linear and Nonlinear DE

Existence and Uniqueness.

The following theorem says that for linear DE $\frac{dy}{dt} = f(t,y)$, every initial value problem has a unique sol'n:

Thm 1. (Fundamental Existence and Uniqueness Theorem (FEUT)):
Let p and g be continuous functions on an open interval $I: \alpha < t < \beta$ containing the point t_0. Then there exists a unique function $y = \varphi(t)$ that satisfies the IVP for all $t \in I$

$$(*) \quad \begin{cases} y' + p(t)\, y = g(t) \\ y(t_0) = y_0 \end{cases} \qquad (*)$$

Notice that this also states that the solution exists on any interval I containing the value t_0 on which the functions p and g are defined.

(*) This is a key idea.

That is, the solution can fail to exist or be discontinuous only at points where at least one of p or g are discontinuous.

Sketch of the proof:
We know from §2.1 that the general sol'n of $(*)$ is given by

$$\mu(t)\, y = \int \mu(t)\, g(t)\, dt + C \qquad (**)$$

where $\mu(t) = e^{\int p(t)\,dt}$

Since $p(t)$ is assumed to be continuous on (α, β), then $\mu(t)$ is also continuous on (α, β). μ is also positive on (α, β)

Since g is continuous on (α, β) the $\mu(t)\,g(t)$ is integrable on (α, β).

To satisfy the initial condition we require that $c = y_0$, and we write the sol'n as

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\,g(s)\,ds + y_0\right]$$

This sol'n exists for all t for which μ and g are defined and continuous. ∎

There is a similar Theorem for nonlinear equations:

<u>Thm 2.</u> Let f and $\partial f/\partial y$ be continuous functions of t and y in some rectangle $(\alpha, \beta) \times (\gamma, \delta)$ containing the point (t_0, y_0). Then, in some interval $t_0 - h_0 < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \varphi(t)$ of the IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

The main difference here is that Thm 2 only ensures that a sol'n exists on <u>some</u> interval containing t_0, not on <u>every</u> interval for which two nice functions are well-defined and continuous.

We won't try to prove Thm 2. You'll do it in an advanced calc or graduate DE course someday.

<u>Ex.</u> Use thm 1 to find an interval on which the IVP has a unique sol'n.

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2. \end{cases}$$

Rewrite as $\begin{cases} y' + \frac{2}{t}y = 4t \\ y(1) = 2 \end{cases}$ so that $p(t) = 2/t$ and $g(t) = 4t$

Now $\text{dom}(p) = (-\infty, 0) \cup (0, \infty)$ and $\text{dom}(g) = \mathbb{R}$

$t=1$ falls in $(0,\infty)$, so the IVP has a unique sol'n on $0 < t < \infty$.

Indeed, we solved this in §2.1 and found the sol'n to be

$$y = t^2 + \frac{1}{t^2}, \quad t > 0.$$

If we change the initial data to be $y(-1) = 2$, then the sol'n would be

$$y = t^2 + \frac{1}{t^2}, \quad t < 0.$$

Ex. Apply Thm 2 to $\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases}$ $(***)$

here $f(x,y) = \dfrac{3x^2+4x+2}{2(y-1)}$ and $\dfrac{\partial f}{\partial y}(x,y) = -\dfrac{(3x^2+4x+2)}{2(y-1)^2}$

These are both continuous for all x, and all $y \neq 1$.

Thus, a rectangle can be drawn around the initial point $(0,-1)$ in which both f and $\partial f/\partial y$ are continuous.
This means the IVP can be solved (uniquely) in some interval around $x=0$, BUT it doesn't say that the sol'n extends over all of $x \in \mathbb{R}$.

We solved this IVP in §2.2 and found the sol'n

$$y = 1 - \sqrt{x^3+2x^2+2x+4}$$

This is only defined for x s.t.

$$\begin{aligned} x^3+2x^2+2x+4 &\geq 0 \\ x^2(x+2)+2(x+2) &\geq 0 \\ (x^2+2)(x+2) &\geq 0 \\ x+2 &\geq 0 \\ x &\geq -2 \end{aligned}$$

but if $x=-2$, then $y=1$, which was not allowed.
Thus, the sol'n to the IVP $(***)$ is only defined for $x > -2$.

These two simple examples that we've already studied highlight the major difference between linear and nonlinear DE; in addition, of course, to the fact that nonlinear can be extremely difficult to solve if they aren't separable.

Ex. Solve the nonlinear DE and determine the interval on which the sol'n exists:

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

$$\frac{dy}{y^2} = dt$$

$$\Rightarrow \quad -\frac{1}{y} = t + C$$

$$-y = \frac{1}{t+C}$$

$$y = \frac{-1}{t+C}$$

Plugging in initial data:

$$y(0) = \frac{-1}{0+C} = 1$$

so $C = -1$

The sol'n is $y = \frac{-1}{t-1} = \frac{1}{1-t}$

Clearly $\lim_{t \to 1^-} y = \infty$, so the sol'n is defined only on $(-\infty, 1)$

What if instead we use initial data $y(0) = y_0$. Then

$$y(0) = \frac{-1}{C} = y_0 \quad \text{and} \quad C = \frac{-1}{y_0}.$$

The sol'n becomes $y = \frac{y_0}{1 - y_0 t}$, and now the sol'n becomes unbounded as $t \to 1/y_0$ from the left. The domain of definition is now $(-\infty, 1/y_0)$.

This shows that the singularities of the sol'n depend on the initial data. (for nonlinear DE)