2.2: Separable Equations

We've already seen some simple examples of this method. Let's look more generally now.

A general first order ODE is given by:

$$\frac{dy}{dx} = f(x,y) \qquad (*)$$

We can arrange that this is given by:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

This can always be done: set $M = -f$ and $N = 1$.

If eq'n $(*)$ can be written in the form:

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (**)$$

then the DE is called <u>separable</u>.

To solve, rearrange and integrate: $\int N(y)\,dy = -\int M(x)\,dx$

then solve for y. (If possible.)

Why does the separate and integrate method work?
let $y = y(x)$ be a function of x and $n = n(y)$ be a function of y.

Then $\frac{d}{dx} n(y) = \frac{dn}{dy} \cdot \frac{dy}{dx} = n'(y) \frac{dy}{dx}$ by the Chain Rule.

If we let $N(y) = n'(y)$, then the DE $(**)$ says

$$\frac{d}{dx} n(y) = -M(x)$$

In other words, $-M$ is an antiderivative of $n(y(x))$ thought of as a function of x. We know from the FTC that we can ~~integrate~~ use indefinite integrals to find anti derivatives. Thus

$$n(y(x)) = n(x,y) = -\int M(x)\,dx.$$

Word.

Ex. $\frac{dy}{dx} = \frac{x^2}{1-y^2} \Rightarrow (1-y^2)\,dy = x^2\,dx$

Integrate: $y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$

or

$$\boxed{y - \frac{1}{3}y^3 - \frac{1}{3}x^3 = C}$$

Each C determines a different integral curve sol'n.

Essentially, we can write a general solution to the IVP

$$\begin{cases} M(x) + N(y)\dfrac{dy}{dx} = 0 \\ y(x_0) = y_0 \end{cases}$$

as

$$\boxed{\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = 0}$$

This solves for the unknown constant as part of the solution to the DE itself.

Ex. Solve the IVP: $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0)=-1$

$$2\int_{-1}^{y} s-1\,ds = \int_{0}^{x} 3s^2+4s+2\,ds$$

$$2\left(\tfrac{1}{2}s^2 - s\right)\Big|_{-1}^{y} = \left(s^3+2s^2+2s\right)\Big|_{0}^{x}$$

$$y^2 - 2y - 3 = x^3 + 2x^2 + 2x \qquad (***)$$

or

$$\boxed{y^2 - 2y - x^3 - 2x^2 - 2x = 3}$$

To solve the IVP explicitly, we still need to solve for y in terms of x; use $(***)$:

$$y^2 - 2y = \underbrace{x^3 + 2x^2 + 2x + 3}_{A} \quad \text{or}$$

$$y^2 - 2y - A = 0$$

Solving for y we get:

$$y = \frac{2 \pm \sqrt{4+4A}}{2} = 1 \pm \sqrt{1+A}$$

so $y = 1 \pm \sqrt{1+x^3+2x^2+2x+3}$

or $y = 1 \pm \sqrt{x^3+2x^2+2x+4}$

Now, the only one that satisfies the DE w/ initial condition $y(0)=-1$ is

$$\boxed{y = 1 - \sqrt{x^3+2x^2+2x+4}}$$! ☺

Ex. (11) $x\,dx + ye^{-x}\,dy = 0$ $y(0)=1$ Solve the IVP.

$$xe^x\,dx + y\,dy = 0$$

$$\int_0^x te^t\,dt + \int_1^y t\,dt = 0$$

$$te^t - e^t \Big|_0^x + \frac{1}{2}t^2 \Big|_1^y = 0$$

$$xe^x - e^x + 1 + \frac{1}{2}y^2 - \frac{1}{2} = 0$$

$$\frac{1}{2}y^2 = e^x - xe^x - \frac{1}{2}$$

$$y^2 = 2e^x - 2xe^x - 1$$

$$y = \pm\sqrt{2e^x - 2xe^x - 1}$$

So the sol'n of the IVP is

☺ $$\boxed{y = \sqrt{2e^x - 2xe^x - 1}}$$

(30!) Ex. "Homogeneous Equations" : $\frac{dy}{dx} = f(x,y)$ and $f(x,y) = g\left(\frac{y}{x}\right)$.

a) Consider the eq'n: $\frac{dy}{dx} = \frac{y-4x}{x-y}$ (*)

divide through by x on top and bottom of RHS:

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - y/x} \qquad (**)$$

b) Introduce a new variable $v = \frac{y}{x}$ so that $y(x) = x\,v(x)$

Then $\frac{dy}{dx} = \frac{d}{dx}\left[x\,v(x)\right] = v(x) + x\,v'(x) = v + xv'$

c) Plug into eq'n (**):

$$v + xv' = \frac{v-4}{1-v}$$

$$xv' = \frac{v-4}{1-v} - v = \frac{v-4-(v-v^2)}{1-v} = \frac{v^2-4}{1-v}$$

so that $x\frac{dv}{dx} = \frac{v^2-4}{1-v}$ or

$$\frac{1-v}{v^2-4}\,dv = \frac{1}{x}\,dx \qquad (***)$$

d) Solve eq'n (***) for v (RE)

e) Sub back in $v = y/x$ into sol'n you found.

f) The expression from e) is a sol'n to eqn (*).
Plot a direction field using a calculator or something.