Chapter 2: First order ODE

In this chapter we'll solve equations like $\frac{dy}{dt} = f(y,t)$.

Any differentiable function $y = \varphi(t)$ that solves this equation for some t is called a solution.

2.1. Integrating Factors

The "separate and integrate" method that we've used so far does not always work in general.

One method, due to Leibniz, is to multiply the entire DE by a function $\mu(t)$ that will make the eq'n easier to integrate. $\mu(t)$ is called an integrating factor. Most of the trouble is in finding the right μ.

Ex. $dy/dt + \frac{1}{2} y = \frac{1}{2} e^{t/3}$

Idea: multiply through by $\mu(t)$:

$$\mu(t) \frac{dy}{dt} + \frac{1}{2}\mu(t) y = \frac{1}{2}\mu(t) e^{t/3} \qquad (*)$$

Question: is the LHS the derivative of a particular expression?

This looks like a product rule of some kind.

Try finding the derivative of the product

$$\frac{d}{dt}\left[\mu(t)\, y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}\, y$$

This looks similar to what we have. For it to be equal

$$\frac{d\mu}{dt} = \frac{1}{2}\mu(t)$$

must hold.

Solve this:

$$\frac{d\mu}{\mu} = \frac{1}{2}\, dt$$

$$\ln \mu = \frac{1}{2}t + C$$

$$\mu = C e^{1/2 t}$$

This is an integrating factor for any C, so let's choose the easy one (1).

Substituting into (*) we have

$$\underbrace{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y}_{\frac{d}{dt}\left[e^{t/2}y\right]} = \frac{1}{2}e^{t/2}e^{t/3}$$

$$\frac{d}{dt}\left[e^{t/2}y\right] = \frac{1}{2}e^{5t/6}$$

To solve: we can now "separate and integrate"

$$\int d\left[e^{t/2}y\right] = \int \frac{1}{2}e^{5t/6}\, dt$$

$$e^{t/2}y = \frac{1}{2}\cdot\frac{6}{5}e^{5t/6} + C$$

$$\boxed{y = \frac{3}{5}e^{t/3} + Ce^{-t/2}}$$

This gives us a <u>general</u> <u>solution</u>!

Suppose we want the <u>particular</u> <u>solution</u> that passes through (0,1).

We set $t=0$ and $y=1$ to obtain

$$1 = 3/5 + C$$

Therefore $C = 2/5$ and the particular sol'n is

$$\boxed{y = \frac{3}{5} e^{t/3} + \frac{2}{5} e^{-t/2}}$$

Pretty rad, right!?

Next, we want a general way to find μ so that we don't have to do <u>so</u> much work each time.

Consider an eqn of the form $\frac{dy}{dt} + ay = g(t)$,
where a is a given constant and $g(t)$ is a known function.

As before, we need μ to satisfy $\frac{d\mu}{dt} = a\mu$
so that $\mu = e^{at}$.

The eqn then becomes

$$\frac{d}{dt}\left[e^{at} y\right] = e^{at} g(t)$$

$$\text{so that} \quad e^{at} y = \int e^{at} g(t)\, dt + C$$

If we can evaluate this integral, then all is good. If $e^{at} g(t)$ is more complicated, then we need to leave the solution in this form:

$$\boxed{y = e^{-at} \int_{t_0}^{t} e^{as} g(s)\, ds + Ce^{-at}}$$

Ex. Solve: $\frac{dy}{dt} - 2y = 4 - t$.

here $a = -2$ so $\mu = e^{-2t}$

The eqn becomes:

$$\frac{d}{dt}\left[e^{-2t} y\right] = (4-t)e^{-2t}$$

$$e^{-2t} y = \int (4-t)e^{-2t}\, dt$$

$$e^{-2t} y = 4\int e^{-2t}\, dt - \int t e^{-2t}\, dt$$

$$= -2e^{-2t} + \tfrac{1}{2} t e^{-2t} - \tfrac{1}{4} e^{-2t} + C$$

$$\text{so} \quad \boxed{y = -\tfrac{7}{4} + \tfrac{1}{2} t + Ce^{2t}}$$

In general a first order linear ODE can be written as

$$\frac{dy}{dt} + p(t)\, y = g(t) \qquad (*)$$

By similar arguments

$$\frac{d\mu}{dt} = p(t)\,\mu$$

must hold, so that $\boxed{\mu = e^{\int p(t)\,dt}}$ is the integrating factor.

The ODE becomes $\frac{d}{dt}\left[\mu(t)\, y\right] = \mu(t)\, g(t)$

which has solution $\mu(t)\, y = \int \mu(t)\, g(t)\, dt + C$

OR

$$\boxed{y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{C}{\mu(t)}} \qquad (**)$$

This is the general form of the solution to this problem.
We can just start from here from now on ☺.

Ex. $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases} \Rightarrow y' + \frac{2}{t}\, y = 4t$

$$\mu = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$$

So the solution is

$$y = \frac{1}{t^2} \int_{t_0}^{t} 4 s^3\, ds + \frac{C}{t^2}$$

$$= \frac{1}{t^2}\left(t^4 - t_0^4\right) + \frac{C}{t^2}$$

$$= t^2 - \frac{t_0^4}{t^2} + \frac{C}{t^2}$$

$$= t^2 + \frac{C - t_0^4}{t^2} = \boxed{t^2 + \frac{C}{t^2}}$$

* Notice: the value of t_0 is not significant at all. It will always be absorbed in the unknown constant.

Now, the initial condition $y(1)=2$ allows us to solve for C:

$$y(1)=2 = 1^2 + \frac{C}{1^2} \Rightarrow C = 2-1 = 1$$

So the particular sol'n is $y = t^2 + \frac{1}{t^2}$, $t>0$.

$t>0$ is necessary because at $t=0$ this equation is undefined. This causes the DE to have separate integral curves for $t>0$ and $t<0$.

Ex. (29) Consider the initial value problem

$$\begin{cases} y' + \frac{1}{4} y = 3 + 2\cos 2t \\ y(0) = 0 \end{cases}$$

Find the solution and describe the behavior as $t \to \infty$.

Sol'n. The int. factor is $\mu = e^{\frac{1}{4}t}$

The sol'n is

$$y = e^{-t/4} \int_{t_0}^{t} 3e^{s/4} + 2e^{s/4} \cos 2s \, ds + ce^{-t/4}$$

Integrate this thing by parts a couple of times.