Math 555: Differential Equations I
Summer 2013

Chapter 1: Introduction

What are differential equations (DE), and why should we study them?

DEs are equations that involve an unknown function and its derivatives. It will be our job to find the function.

DEs are used to model physical phenomena such as:

- fluid dynamics (motion)
- electrical flow
- dissipation of heat
- propogation of seismic waves
- population growth, etc.

Example. Population growth/decay.

Consider a pop'n of field mice. If there are no predators, assume that the pop'n increases at a rate proportional to its size. i.e., if p = pop'n, then

$$\frac{dp}{dt} = rp$$

where r is constant and t is time. Suppose t is measured in months.

Also suppose there are owls in the region who kill 15 mice per day, or ~450 mice / mo.

k mice / mo.

Then the monthly growth eqn becomes

$$\frac{dp}{dt} = rp - k = r\left(p - k/r\right)$$

We want to find an equation for $p = p(t)$. To do this, we separate and integrate.

$$\frac{dp}{p - k/r} = r\,dt$$

$$\ln\left(p - k/r\right) = rt + C$$

$$p - k/r = C e^{rt}$$

$$p = Ce^{rt} + k/r$$

To find the constant C we need to know the pop'n at a given time. For example, $p(0) = P_0$.

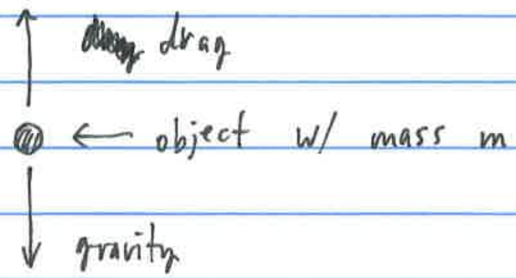
Then $p(0) = Ce^{0} + k/r = C + k/r$, so $C = P_0 - k/r$

The solution of the predator/prey DE is now:

$$p(t) = \left(P_0 - k/r\right)e^{rt} + k/r$$

Notice: if $P_0 = k/r$ then $p(t) = k/r$ is constant.
This is the equilibrium solution.

__Example.__ An object is falling near sea level.
Find a DE that describes the motion.



Let g be the acceleration due to gravity, so that the force due to gravity is mg by Newton's second law. ($F=ma$).

We'll also assume that drag is proportional to velocity:

$$\text{drag} = \gamma v \quad \text{for some constant } \gamma.$$

Moreover, acceleration $a = dv/dt$ is the derivative of velocity.

By Newton's 2^{nd} Law again, we write

$$F = ma = m\frac{dv}{dt}$$

but by our figure above $F = \text{gravity} - \text{drag}$

$$= mg - \gamma v$$

$F=F$ (obviously) so our DE is

$$m\frac{dv}{dt} = mg - \gamma v \quad \text{or}$$

$$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m} v}$$

solve this as in the first example.

Now suppose $m = 10\ kg$, $r = 2\ kg/s$, and $g = 9.8\ m/s^2$.

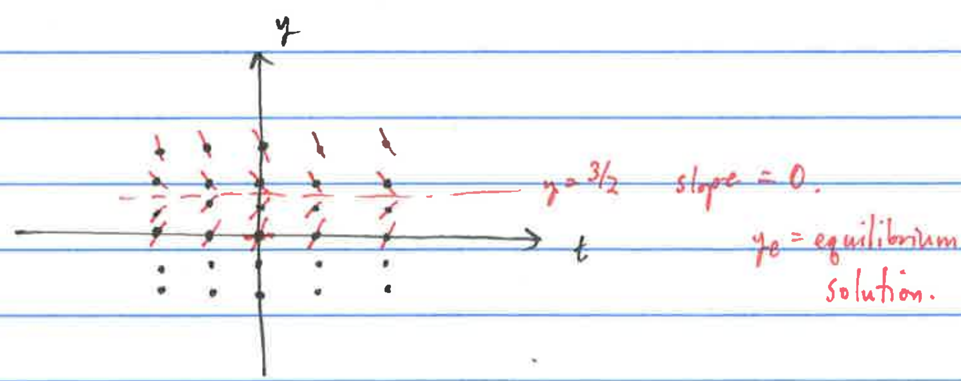
Then our DE is $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

This is an equation for the slope $\left(\frac{dv}{dt}\right)$ of the function v at the points $(t, v(t))$.
We can draw a slope field to try to visualize what solutions of this equation look like.

Example. [1] draw a slope field for the DE

$$y' = 3 - 2y$$

the solve the DE.



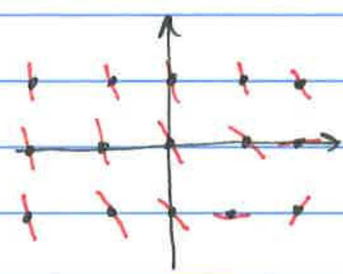
From any starting point, as $t \to \infty$, all solutions approach $y = 3/2$, the equilibrium sol'n.

Example. (8) Consider $\frac{dy}{dt} = ay + b$

Find an eqn in which all solutions approach $y = 2/3$

two choices: $\underbrace{a=3, \quad b=-2}_{\text{diverge from}}$ or $\boxed{\underbrace{a=-3, \quad b=2}_{\text{approach}}}$

Example (26) Draw a slope field for $y' = -2 + t - y$



Example. Consider the DE $\frac{dy}{dt} = ay - b$ $(*)$

$$\frac{dy}{y - b/a} = a\,dt$$

$$\ln\left(y - b/a\right) = at + C$$

$$y - b/a = Ce^{at}$$

$$\boxed{y = Ce^{at} + b/a}$$

This is called the general solution of the DE $(*)$. It represents a family of curves called integral curves. There is one integral curve for each value of C.

<u>Example.</u> (7) Recall our field mouse equation

$$\frac{dp}{dt} = rp - 450$$

The sol'n was $p(t) = \left(p_0 - \frac{450}{r}\right) e^{rt} + \frac{450}{r}$

If we assume $r = 1/2$, then this solution becomes

$$p(t) = (p_0 - 900)\, e^{t/2} + 900$$

a) Find the time that the mice become extinct if $p_0 = 850$

$$p(t) = -50\, e^{t/2} + 900 = 0$$

$$e^{t/2} = \frac{900}{50} = 18$$

$$t/2 = \ln 18$$

$$t = 2 \ln 18 \approx 5.78 \text{ mo.}$$

b) Find the time of extinction if $0 < p_0 < 900$.

$$(p_0 - 900)\, e^{t/2} = -900$$

$$e^{t/2} = \frac{-900}{p_0 - 900}$$

$$t/2 = \ln\left(\frac{-900}{p_0 - 900}\right)$$

$$t = 2 \ln\left(-900/p_0 - 900\right)$$

c) Find the initial pop'n if the pop'n is to be extinct in 1 yr.

$$p(12) = 0$$

$$p(12) = (p_0 - 900)e^6 + 900$$

$$p_0 - 900 = \frac{-900}{e^6}$$

$$p_0 = 900 - \frac{900}{e^6}$$

$$p_0 \approx 897.77 \approx 898$$

Classifying DEs:

Ordinary DE: The unknown function is of one variable and the equation relates it and its derivatives.

Partial DE: The function is of multiple variables and the equation relates it and its partial derivatives.

Examples of PDE:

heat eqn: $$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

wave eqn: $$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

We won't study PDE in this course.

An ODE:
$$L \frac{d^2 Q(t)}{dt^2} + R \frac{dQ(t)}{dt} + \frac{1}{C} Q(t) = E(t)$$

$Q(t)$ = charge on a capacitor in a circuit w/ capacitance C, resistance R, inductance L.

The order of a DE is the highest derivative that appears in the equation.

More generally: $F[t, u(t), u'(t), \ldots, u^{(n)}(t)] = 0$
is an ODE of order n. or
$$F(t, y, y', \ldots, y^{(n)}) = 0.$$

We will always assume that we can solve a DE for its highest derivative, so that

$$F(t, y, y', \dots, y^{(n)}) = 0$$

can be rewritten as

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}).$$

We'll only study equations of this form.

Linear vs. Nonlinear ODE.

An ODE is linear if the eq'n

$$F(t, y, y', \dots, y^{(n)}) = 0$$

is a linear ~~equation~~ equation; i.e., if F is a linear function of $t, y, y', \dots, y^{(n)}$.

A general linear ODE of order n looks like:

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_n(t)\, y = g(t).$$

Most (if not all) of the DE that we will study are linear.

Solutions. A solution to the DE $F(t, y, \dots, y^{(n)}) = 0$ is a function $\varphi(t)$, $\alpha < t < \beta$ that satisfies

$$\varphi^{(n)}(t) = f(t, \varphi(t), \dots, \varphi^{(n-1)}(t))$$

for all $\alpha < t < \beta$.

It's not always easy to find a solution to a given DE, but given a function and a DE, it's easy to check if the function is in fact a sol'n.

Example (2). Classify the DE: $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$

2nd order nonlinear ODE.

Example (8). verify that $y(t) = e^{-3t}$ is a solution of the DE:

$$y'' + 2y' - 3y = 0$$

$$y'(t) = -3e^{-3t}$$
$$y''(t) = 9e^{-3t}$$

Plugging in: $9e^{-3t} + 2(-3e^{-3t}) - 3e^{-3t} = 9e^{-3t} - 9e^{-3t} = 0.$ ✓

Example (16). Find all values of r such that $y = e^{rt}$ is a sol'n of

$$y'' - y = 0$$

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

$$r^2 e^{rt} - e^{rt} = 0$$
$$e^{rt}(r^2 - 1) = 0$$
$$r^2 - 1 = 0$$
$$\boxed{r = \pm 1}$$