Honors DE : Assignment III

Due by 13 December 2013

In this assignment we'll take a closer look at some theory behind the numerical methods for approximating solutions to first order DE that we have studied in class.

Euler's Method

We begin by showing that the approximate solutions found using Euler's method actually converge to the solution of the IVP as h tends to 0.

Consider the first order initial value problem

$$\begin{cases} y' = f(t, y)\,; \\ y(t_0) = y_0. \end{cases} \tag{1}$$

Suppose that f is Lipschitz continuous. (Look back at Assignment I if you need a refresher on Lipschitz continuity.) Moreover, suppose that the exact solution $y = \varphi(t)$ is C^2 on the domain which it is defined. This means that φ is continuous, and φ', φ'' exist and are continuous on the domain of definition. Under these conditions on f and φ, it can be shown that Euler's method converges to the actual solution of the IVP (1) on the domain of definition.

Requiring that f be Lipschitz ensures that the tangent lines used to piece together the approximate solution do not become "too vertical". Insisting that φ is C^2 ensures that the solution we are hoping to converge to is smooth enough that it can actually be approximated by tangent lines. The proof of this theorem is a little beyond the scope of this course, but the following exercise give an example for which Euler's method converges.

Ex 1 Consider the initial value problem

$$\begin{cases} y' = 1 - t + y\,; \\ y(t_0) = y_0. \end{cases} \tag{2}$$

1. Show that for any initial data (t_0, y_0), the domain of the solution $y = \varphi(t)$ is $\mathbb{R}$.
2. Show that the exact solution is $y = \varphi(t) = (y_0 - t_0)e^{t-t_0} + t$.
3. Using Euler's formula $y_{k+1} = y_k + f_k \cdot h$, show that
$$y_{k+1} = (1+h)y_k + h - ht_k, \quad k = 0, 1, 2, \ldots$$
4. Noting that $y_1 = (1+h)(y_0 - t_0) + t_1$, show by induction that
$$y_n = (1+h)^n(y_0 - t_0) + t_n \tag{3}$$
for all $n = 1, 2, 3, \ldots$.
5. Consider a fixed point $t^* > t_0$ and for a given n choose $h = \frac{1}{n}(t^* - t_0)$. Then $t_n = t^*$ for every n. Note that $h \to 0$ as $n \to \infty$. By substituting for h in equation (3) and letting $n \to \infty$, show that $y_n \to \varphi(t^*)$. [Hint: $\lim\limits_{n\to\infty}(1 + \frac{a}{n})^n = e^a$.]

Since $t^* > t_0$ was arbitrary, you have shown that Euler's method converges to $\varphi(t)$ for all $t > t_0$. That is, we have only recovered the right-hand side of the exact solution. The other "half" of the solution can be recovered by making appropriate changes to Euler's formula.

Ex 2 Write down an analogue to Euler's method that will approximate solutions to the left of the initial data.

You could now repeat **Ex 1** to show that Euler's method converges on the entire domain of definition of φ, and therefore recovers all of φ, but I won't make you do that.

A Better Euler Method

Once again consider the IVP (1). If $y = \varphi(t)$ is the exact solution, and $\{t_0, \ldots, t_n\}$ is a partition of an interval $[t_0, t_n]$ with $t_{k+1} - t_k = h$ for all k, then φ satisfies the integral equation
$$\varphi(t_{k+1}) = \varphi(t_k) + \int_{t_k}^{t_{k+1}} f(t, \varphi(t))\, dt. \tag{4}$$
This formula should look familiar as it comes from the Picard method that we used to prove the FEUT in assignment I.

The Euler formula
$$y_{k+1} = y_k + h \cdot f(t_k, y_k) \tag{5}$$
is obtained from equation (4) by replacing $f(t, \varphi(t))$ by its value at the left endpoint, $f(t_k, y_k)$. Therefore, Euler's method can be thought of as the left endpoint rule for approximate integration that you studied in Calc II.

The connection can be seen by rearranging the terms in each of the last two equations to obtain

$$\varphi(t_{k+1}) - \varphi(t_k) = \int_{t_k}^{t_{k+1}} f(t, \varphi(t))\, dt \approx h \cdot f(t_k, y_k). \tag{6}$$

As written, this formula really only makes sense for $k = 0$, but the idea is exactly the same for all other iterations. Again thinking back to Calc II, we know that we can get a better approximation of this integral by taking the area of trapezoids instead of rectangles. We thus obtain a new formula for updating the value y_{k+1},

$$y_{k+1} = y_k + h \cdot \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}. \tag{7}$$

The problem with this formula is that y_{k+1} shows up on the right-hand side, but we don't know what y_{k+1} is; that's what we're trying to find!

To get around this issue, we can use the old Euler formula (5) to approximate the y_{k+1} on the right-hand side of equation (7). We obtain

$$y_{k+1} = y_k + h \cdot \frac{f(t_k, y_k) + f(t_{k+1}, y_k + h \cdot f(t_k, y_k))}{2}. \tag{8}$$

This formula is called the *improved Euler formula*, or the *Heun formula*.

Ex 3 Use the improved Euler method to approximate solutions to the IVP (2) with initial data $y(0) = 2$ on the interval $[0, 2]$. Use step sizes of $h = 0.1$ and $h = 0.05$. Also use the old Euler method to solve the same problem, and compare your results.

As a final exercise, let's take a look at the error generated in this improved Euler method. If such a method is to be used in practice, then we need to know how small we have to choose h to get an approximation that is "close enough" to the actual solution. On the other hand, we don't want to do more work than we absolutely need to. Thus we should choose the optimal value for h that gives us the accuracy we want, but doesn't make us do any more work than we need to.

Suppose that the exact solution is C^3. This is very similar to the C^2 condition that in the first section, except that we must also add φ''' into the mix. We want to show that the local truncation error for the improved Euler method is proportional to h^3. By assuming that φ is C^3, we are able to write a Taylor expansion for φ around t_k up to third order. We get

$$\varphi(t_{k+1}) = \varphi(t_k + h) = \varphi(t_k) + \varphi'(t_k) \cdot h + \frac{\varphi''(t_k)}{2!} \cdot h^2 + \frac{\varphi'''(t_k^*)}{3!} \cdot h^3, \tag{9}$$

where $t_k < t_k^* < t_{k+1}$. Now assume that $y_k = \varphi(t_k)$.

Ex 4 Show that the error generated by the $(k+1)$-st step is given by

$$\begin{aligned} e_{k+1} &= \varphi(t_{k+1}) - y_{k+1} \\ &= \frac{\varphi''(t_k) - [f(t_{k+1}, y_k + h \cdot f_k) - f_k]}{2!} \cdot h + \frac{\varphi'''(t_k^*)}{3!} \cdot h^3 \end{aligned}$$

by plugging in for y_k using equation (8).

Ex 5 Use the facts that $\varphi'' = f_t(t, \varphi(t)) + f_y(t, \varphi(t))\varphi'(t)$ and the Taylor approximation with a remainder for a function $F(t, y)$ of two variables is

$$F(a+h, b+k) = F(a,b) + F_t(a,b) \cdot h + F_y(a,b) \cdot k + \frac{1}{2!} \left[h^2 F_{tt} + 2hkF_{ty} + k^2 F_{yy}\right] \Big|_{(\xi,\eta)},$$

where $\xi \in (a, a+h)$ and $\eta \in (b, b+k)$, show that the first term on the right-hand side of the equation for e_{k+1} is proportional to h^3 plus higher order terms. This is the desired result.

Ex 6 Show that if $f(t, y)$ is linear in both t and y, then $e_{k+1} = \dfrac{h^3 \varphi'''(t_k^*)}{6}$ for some $t_k^* \in (t_k, t_{k+1})$. [Hint: What are f_{tt}, f_{ty}, and f_{yy}?]