Honors DE : Assignment II

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Let's take a little bit closer look at power series. We will derive two very important classes of polynomials known as the *Chebyshev polynomials* and *Legendre polynomials*. In each case, the first n such polynomials form a basis for the vector space P_n of $(n-1)$-degree polynomials. Then we'll show exactly how the coefficients a_n of a general power series solution of a general second order DE can always be written in terms of a_0 and a_1.

1 Chebyshev Polynomials

Consider the differential equation

$$(1-x^2)y'' - xy' + \alpha^2 y = 0.$$

This equation is called *Chebyshev's equation*. It has two singular points, $x = \pm 1$. Therefore, we know that series solutions around $x_0 = 0$ will have a radii of convergence of at least 1. (Why?)

Ex 1 Find the first four terms of each of y_1 and y_2 centered at $x_0 = 0$, and show that they form a fundamental solution set.

If $\alpha = n$ is a non-negative integer, *i.e.,* if $\alpha = 0, 1, 2, 3, 4, \ldots$, then one of the two fundamental solutions will become a (finite) polynomial of degree n. These are called the *Chebyshev polynomials*. They are defined on the interval $[-1, 1]$.

Ex 2 Consider the initial data $y(0) = 1$ and $y'(0) = 1$. Find the Chebyshev polynomials that arise from this initial value problem for $\alpha = 0, 1, \ldots, 7$. Call these polynomials $C_0, C_1, \ldots, C_7$.

Ex 3 Graph each of the polynomials that you found in the last **Ex** in a different color on the interval $[-1, 1]$. Do you notice anything interesting about the behavior of the evens and odds?

The Chebyshev polynomials that you just found form a basis for the vector space $P_8[-1, 1]$, the 7^{th} degree (and below) polynomials defined on the interval $[-1, 1]$. This means that any polynomial in P_8 can be written as a linear combination of the Chebyshev polynomials. Moreover, all of the odd-degree polynomials satisfy

$$\int_{-1}^{1} C_{2k+1}(x)\, dx = 0.$$

Ex 4 Calculate the definite integrals $I_{2k} := \int_{-1}^{1} C_{2k}(x)\,dx$ for $k = 0, 1, 2, 3$.

Now the integral of any polynomial in $P_8[-1, 1]$ can be calculated using only the numbers I_0, I_2, I_4, and I_6.

Ex 5 Write the polynomial

$$p(x) = 32x^6 + 16x^5 - 64x^4 - 20x^3 + 40x^2 + 4x - 3$$

as a linear combination of C_n's. Write $\int_{-1}^{1} p(x)\,dx$ as a linear combination of I_n's, and calculate the exact value.

We can also use these numbers I_n to approximate the integrals of sufficiently smooth functions on the interval $[-1, 1]$.

Ex 6 Write the 6^{th} degree Taylor polynomials for e^x and $\cos(x)$ around the point $x_0 = 0$. Write the even terms for each as linear combinations of C_0, C_2, C_4, and C_6. (This might be messy, sorry.) Estimate the integrals $\int_{-1}^{1} e^x\,dx$ and $\int_{-1}^{1} \cos(x)\,dx$ in terms using I_0, I_2, I_4, and I_6. How close are the answers?

2 Legendre Polynomials

Legendre's equation is given by

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

Like Chebyshev's equation, it has singular points at $x = \pm 1$. Again, this means that we can look for power series solutions in the interval $[-1, 1]$, and be sure that they will converge. Also like Chebyshev's equation, if $\alpha = n$ is a non-negative integer, then one of the two solutions of this equation will become a polynomial of degree n. The *Legendre polynomials of degree* n, denoted by L_n, are the polynomial solutions with the additional constraint that $L_n(1) = 1$.

Ex 7 Find the Legendre polynomials $L_0, L_1, \ldots, L_7$. Plot each of them in a different color on the interval $[-1, 1]$. Find the zeros of each L_n in the interval. (You can use a CAS to help with this.)

Ex 8 Show that Legendre's equation can also be written as

$$[(1 - x^2)y']' = -\alpha(\alpha + 1)y.$$

The last **Ex** implies that

$$[(1 - x^2)L_n']' = -n(n + 1)L_n, \text{ and} \tag{1}$$

$$[(1 - x^2)L_m']' = -m(m + 1)L_m. \tag{2}$$

Ex 9 Multiply equation (1) by L_m and equation (2) by L_n, integrate by parts, then subtract one equation from the other to show that

$$\int_{-1}^{1} L_n(x)L_m(x)\,dx = 0 \tag{3}$$

for $m \neq n$.

Equation (3) is called an *orthogonality property*. You may remember from Linear Algebra that if a vector space has an inner product $\langle\ ,\ \rangle$, then two vectors x and y are called orthogonal iff $\langle x, y\rangle = 0$. It can be shown that the integral in equation (3) defines an inner product on the vector space $P_8[-1,1]$. For any two polynomials p and q in $P_8[-1,1]$,

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

It can further be show that if $m = n$, then

$$\langle L_n, L_n\rangle = \int_{-1}^{1} L_n^2(x)\,dx = \frac{2}{2n+1}.$$

From these facts, you can deduce that polynomials of the form $\frac{\sqrt{2n+1}}{\sqrt{2}}L_n$ form an orthonormal basis for $P_8[-1,1]$.

3 General Series Near an Ordinary Point

Consider a second order DE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4}$$

where P, Q, and R are analytic functions. Remember, a function is analytic if it can be represented as a power series around every point in its domain. Now assume that equation (4) has a power series solution around the point x_0,

$$y = \varphi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

and that φ converges for all $|x - x_0| < R$, for some $R > 0$. For all of the examples that we did in class, the solution φ could be separated into two separate power series,

$$\varphi = a_0 y_1 + a_1 y_2,$$

where y_1 and y_2 form a fundamental solution set. It turns out that this always the general case, and not just special to our examples. What we need to show is that every a_n can be written in terms of a_0 and a_1.

Inside the interval of convergence for φ, equation (4) can be rewritten as

$$\varphi''(x) = -p(x)\varphi'(x) - q(x)\varphi(x),$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Plugging in $x = x_0$ picks out the constant term of each series for φ, so $\varphi''(x_0) = 2!a_2$. But by the last equation,

$$\varphi''(x_0) = -p(x_0)\varphi'(x_0) - q(x_0)\varphi(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Putting it all together,

$$a_2 = \frac{1}{2!}\left[-p(x_0)a_1 - q(x_0)a_0\right].$$

Ex 10 Differentiate $\varphi''(x) = -p(x)\varphi'(x) - q(x)\varphi(x)$ and rearrange to obtain an expression for φ''' in terms of φ'', φ', and φ. Plug in $x = x_0$ to obtain a_3 in terms of a_2, a_1, and a_0. Substitute in the formula we just found for a_2 to get a_3 solely in terms of a_0 and a_1.

Ex 11 Repeat the method of the last **Ex** to find a_4 in terms of a_0 and a_1.

This method can be repeated as many times as necessary to obtain a generic expression for any a_n in terms of p, q, a_0, and a_1.

Ex 12 Use the formulas for $\varphi^{(n)}$, $n = 0, \ldots, 4$ from the previous results to complete problems 1 and 2 on page 269 of our text. Since you put in the hard work to derive those formulas, these problems should just be simple plug and chug exercises.

Now you could go back and do the same for the Chebyshev and Legendre equations, and derive the first few polynomials using this method instead. You should get the exact same answers.