Honors DE : Assignment I

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This assignment is meant to help you better understand the new analysis concepts used in the proof of the FEUT in Chapter 17 of [3]. In our class we are only studying 1-dimensional "systems" of DE, so every time the book says $\mathbb{R}^n$ you should read $\mathbb{R}$ (or $\mathbb{R}^1$), and every time there is a vector X you should think of it as a single function y.

The most important part of this assignment is for you to read through Chapter 17 of [3] carefully, and understand what it is saying. There are some exercises labeled with **Ex** in these notes that you should try to complete as you go along. If you are stuck or have any questions at all, you should ask me. This is not a test, it's meant to help you *learn* the material.

Suprema and Infima

Definition 1 Let S be a subset of the real numbers $\mathbb{R}$.

1. An element $u \in \mathbb{R}$ is said to be an *upper bound* of S if $s \leq u$ for all $s \in S$.
2. An element $w \in \mathbb{R}$ is said to be a *lower bound* for S if $w \leq s$ for all $s \in S$.

It is possible that a subset S may not have either an upper or a lower bound.

Definition 2 Let S be a subset of $\mathbb{R}$.

1. If S is bounded above, then an upper bound is said to be a *supremum* if it is less than any other upper bound of S. A supremum is also called a least upper bound, or a *lub*.
2. If S is bounded below, then a lower bound is said to be an *infimum* if it is greater than any other lower bound of S. An infimum is also called a greatest lower bound, or a *glub*.

If the set S has a maximum, then the supremum coincides with the max. Likewise, if S has a minimum, then the infimum coincides with the minimum.

Ex 1 Consider the set $S = \left\{1 + \frac{1}{n}\right\}$, $n = 1, 2, 3, \ldots$. Find the sup and inf of S. Is either one a max or min?

Ex 2 Give an example of a set of rational numbers that is bounded but does not have a rational supremum.

Ex 3 Give an example of a set of irrational numbers that has a rational infimum.

Lipshitz Continuity

Definition 3 A function $f : (\alpha, \beta) \to \mathbb{R}$ is said to be *Lipshitz continuous* (or just Lipshitz) if there is a constant $K \geq 0$ such that

$$|f(x) - f(y)| \leq K\,|x - y|$$

for all x and y in (α, β). The constant K is called the *Lipshitz constant* for f.

Notice that the interval $I = (\alpha, \beta)$ is assumed to be *open* in this definition; that is, it doesn't include its end points. Frequently we can take $I = \mathbb{R}$.

Definition 4 Let $f : (\alpha, \beta) \to \mathbb{R}$ be a function such that for every point x in (α, β) there is a constant $\varepsilon = \varepsilon(x)$ such that $I_\varepsilon := (x - \varepsilon, x + \varepsilon)$ is a subset of (α, β) and f is Lipshitz on the interval I_ε. Then f is said to be *locally Lipshitz.*

In this case, for each $x \in (\alpha, \beta)$, f satisfies

$$|f(y_1) - f(y_2)| \leq K_\varepsilon\,|y_1 - y_2|$$

for all y_1 and y_2 in I_ε. In particular, there could be a different Lipshitz constant K_ε for each subinterval I_ε.

Ex 4 Draw a schematic picture that includes the open interval $I = (\alpha, \beta)$, a point $x \in I$, the subinterval I_ε around x, and the graph of f over I_ε. Illustrate how the Lipshitz condition restricts how the graph of f is allowed to behave over I_ε.

Ex 5 Prove that if $f : (\alpha, \beta) \to \mathbb{R}$ is Lipshitz, then it is also locally Lipshitz.

Ex 6 Is the function $f(x) = \sin(x)$ Lipshitz? If so, find K.

Ex 7 Show that the function $f(x) = e^x$ is not Lipshitz, but is locally Lipshitz. Find K_ε in terms of x and ε.

Proving the Lemmas

You should now be able to read and understand the proof of the Lemma on page 387 of [3]. Here are a couple of things to keep in mind that should make it easier to read:

1. Replace $F : \mathcal{O} \to \mathbb{R}^n$ with $f : (\alpha, \beta) \to \mathbb{R}$, as we only care about the 1-dimensional case right now.
2. When you see DF_X read $\frac{df}{dx}(x)$, where these x's are confusingly different. The x in dx is the independent variable while the x in (x) is the point in (α, β) where the derivative is to be evaluated.

3. A set is *convex* if every pair of points can be connected by a straight line that sits entirely in the set. For example, a disc is convex, but a "fat" letter U is not. In the 1-dimensional case, every connected interval (without holes or breaks) is convex, so this is not a crazy assumption in any way.

After you believe the proof of the lemma, read the first paragraph on page 388 carefully. This is a key idea in the proof of the FEUT. We actually discussed this in class in some detail. It says that we can replace the *differential* equation

$$y' = f(y); \quad y(0) = y_0$$

with the *integral* equation

$$y(t) = \int_0^t f(y(s))\, ds + y_0.$$

Any function y that solves the DE will necessarily solve the integral equation as well. Unfortunately, these equations are not "equivalent": There are functions that satisfy the integral equation but do *not* solve the DE.

Before we continue, we should rewrite the four assumptions in the middle of the page.

1. $I_\varepsilon = [y_0 - \varepsilon, y_0 + \varepsilon]$ is a closed interval of radius ε centered at y_0.

2. There is a Lipshitz constant K_ε for f on I_ε.

3. $|f(y)| \le M_\varepsilon$ on I_ε; *i.e.*, f is bounded on I_ε.

4. Choose $a < \min\left\{\frac{\varepsilon}{M_\varepsilon}, \frac{1}{K_\varepsilon}\right\}$ and put $J = [-a, a]$.

The functions $U_0, U_1, \ldots$ in the next paragraph are going to be the Picard iterations $\varphi_0, \varphi_1, \ldots$ that we studied in class. The "Lemma from Analysis" that follows is a fundamental piece of the puzzle.

Uniform Convergence

Rewrite the lemma as follows:

Lemma 5 *Suppose $\varphi_k : J \to \mathbb{R}$, $k = 0, 1, 2, \ldots$, is a sequence of continuous functions defined on the closed interval J that satisfy:*

Given $\varepsilon > 0$, there is some $N > 0$ such that for every $p, q > N$

$$\max_{t \in J} |\varphi_p(t) - \varphi_q(t)| < \varepsilon.$$

Then there is a continuous function $\varphi : J \to \mathbb{R}$ such that

$$\lim_{k \to \infty} \left\{ \max_{t \in J} |\varphi_k(t) - \varphi(t)| \right\} = 0.$$

Moreover, for any t with $|t| < a$,

$$\lim_{k \to \infty} \int_0^t \varphi_k(s)\, ds = \int_0^t \varphi(s)\, ds.$$

In other words, this says that the sequence of functions φ_k converges to a limit function φ, and that the integrals also converge. You can probably guess that this is exactly what we will need to prove that the Picard iterations converge to an actual solution of the DE.

Ex 8 For every $n = 1, 2, 3, \ldots$, define $f_n(x) = \frac{x}{n}$ and define $f(x) = 0$ for all $x \in \mathbb{R}$.

1. Show that the sequence $\{f_n\}$ satisfies the hypotheses of the lemma (except that the sequence starts at 1 instead of 0, but that's not a big deal).
2. Use a computer to graph the first few functions in the sequence on the interval $[-2, 2]$.
3. Show that $\{f_n\}$ converges uniformly to f for all $x \in \mathbb{R}$ by showing that $|f_n(x) - f(x)| < \varepsilon(n)$ for all n and $\varepsilon(n) \to 0$ as $n \to \infty$. What is $\varepsilon(n)$?

The Proof

You should now be able to work through all of the calculations and claims in the proof starting near the top of page 389 and continuing to the bottom of page 391. Every time a claim is made, identify what assumption or analysis fact it is a result of. Then verify that you believe that the claim is justified and the conclusion is correct.

You should rewrite all of the calculations in the proof yourself as you work through it. Replace all U's with φ's, X's with y's, F's with f's, and ρ's with ε's. If you get stuck on anything (like maybe Induction?), let me know and I will try to help.

Good luck, and enjoy!

References

[1] R. Bartle, *The Elements of Real Analysis.* New York: John Wiley and Sons, 1976.

[2] M. Hirsh and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra.* London: Academic Press, 1974.

[3] M. Hirsh, S. Smale, and R. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* San Diego: Academic Press, 2004.
http://www.math.upatras.gr/~bountis/files/def-eq.pdf