Math 555: Differential Equations I
Final Exam

Monday, 9 December 2013

Name: KEY

Instructions: Complete all 4 problems in part I, and 4 of the 5 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 10 points. The remaining 20 points come from the Final Project.

Show all work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use three (3) 3×5 index cards of your own notes, a pencil, and your brain.

Good Luck!

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Part I: *Complete all 4 problems.*

! **1. True or False:** Read each statement *carefully*, then write **T** or **F** in the space provided. Each is worth 2 points.

F a.) If y_1 and y_2 are any two solutions of the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0,$$

then the Wronskian $W(y_1, y_2)(t_0) \neq 0$. Might not be fund. soln. set

T b.) The general solution of the first order linear initial value problem

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{y_0}{\mu(t)},$$

where $\mu(t) = e^{\int p(t)dt}$.

! T c.) If ~~the~~ a first order differential equation $y' = f(t, y)$ is separable, then it is exact.

F d.) If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval (a, b), then the Wronksian is given by

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$$W(y_1, y_2)(t) = Ce^{\int p(t)dt}, \quad = Ce^{-\int p(t)dt}$$

where C is a constant that depends on y_1 and y_2, but not on t.

T e.) If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$F(s - c) = \mathcal{L}\left\{e^{ct}f(t)\right\}, \quad s > a + c.$$

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2. Find the general solution of the differential equation.

$$y'' - 2y' + y = \frac{2e^t}{1+t^2}$$

Homog: $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$\Rightarrow \quad y_1 = e^t, \quad y_2 = te^t$

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} = (1+t)e^{2t} - te^{2t} = e^{2t}$$

Variation of Parameters:

$$y = y_1(t)\int_{t_0}^{t} \frac{-y_2(s)g(s)}{W(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}\,ds$$

$$= e^t \int_{t_0}^{t} \frac{-se^s(2e^s)}{(1+s^2)e^{2s}}\,ds + te^t \int_{t_0}^{t} \frac{e^s(2e^s)}{(1+s^2)e^{2s}}\,ds$$

$$y = -e^t \ln(1+t^2) + c_1 e^t + 2te^t \tan^{-1} t + C_2 te^t$$

$$\boxed{y = C_1 e^t + C_2 te^t - e^t \ln(1+t^2) + 2te^t \tan^{-1}(t)}$$

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3. Consider the differential equation

$$y'' - 2y' + 2y = 3e^t + 10t^2 - 12t + 18 \qquad (*)$$

3 *a.)* Find the solution of the corresponding homogeneous problem.

$y'' - 2y' + 2y = 0$

$r^2 - 2r + 1 = -1$

$(r-1)^2 = -1$

$r - 1 = \pm i$

$r = +1 \pm i$

thus, $\boxed{y_h = e^{+t}(c_1 \cos t + c_2 \sin t)}$

2 *b.)* Write the form of the particular solution $Y(t)$ of equation $(*)$, if you were to use the method of undetermined coefficients.

$\boxed{Y(t) = Ae^t + Bt^2 + Ct + D}$

5 *c.)* Find the general solution of equation $(*)$.

$Y'(t) = Ae^t + 2Bt + C$

$Y''(t) = Ae^t + 2B$

$Y'' - 2Y' + 2Y = Ae^t - 2Ae^t + 2Ae^t + 2B - 4Bt - 2C + 2Bt^2 + 2Ct + 2D$

$= 3e^t + 10t^2 - 12t + 18$

e^t: $A = 3$

t^2: $2B = 10 \Rightarrow B = 5$

t: $-4B + 2C = -12 \Rightarrow -20 + 2C = -12 \Rightarrow C = 4$

#: $2B - 2C + 2D = 18 \Rightarrow 10 - 8 + 2D = 18 \Rightarrow D = 8$

$Y(t) = 3e^t + 5t^2 + 4t + 8$

General sol'n: $\boxed{y = c_1 e^{+t}\cos t + c_2 e^{+t}\sin t + 3e^t + 5t^2 + 4t + 8}$

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4. Use the <u>*definition*</u> of the Laplace transform to find $\mathcal{L}\{f(t)\}$, where $f(t) = \sin(4t)$.

$$\mathcal{L}\{\sin(4t)\} = \int_0^\infty \sin(4t)\, e^{-st}\, dt = \lim_{A\to\infty}\left[\int_0^A \sin(4t)\, e^{-st}\, dt\right]$$

First:

$$\int \sin(4t)\, e^{-st}\, dt = \int \sin(4t)\left[\frac{d}{dt}\left(\frac{-1}{s}e^{-st}\right)\right] dt = -\frac{1}{s}e^{-st}\sin(4t) + \frac{4}{s}\int \cos(4t)\left[\frac{d}{dt}\left(\frac{-1}{s}e^{-st}\right)\right]dt$$

$$= \frac{-1}{s}e^{-st}\sin(4t) + \frac{4}{s}\left[\frac{-1}{s}e^{-st}\cos(4t) - \frac{4}{s}\int \sin(4t)\, e^{-st}\, dt\right]$$

$$= -\frac{1}{s}e^{-st}\sin(4t) - \frac{4}{s^2}e^{-st}\cos(4t) - \frac{16}{s^2}\int \sin(4t)\, e^{-st}\, dt$$

Then

$$\frac{s^2+16}{s^2}\int \sin(4t)\, e^{-st}\, dt = \frac{-1}{s}e^{-st}\sin(4t) - \frac{4}{s^2}e^{-st}\cos(4t)$$

$$\int \sin(4t)\, e^{-st}\, dt = \frac{s^2}{s^2+16}\left[\frac{-1}{s}e^{-st}\sin(4t) - \frac{4}{s^2}e^{-st}\cos(4t)\right]$$

Plug in $t=0, t=A$ limits:

$$\int_0^A \sin(4t)\, e^{-st}\, dt = \frac{s^2}{s^2+16}\left[\frac{-1}{s}e^{-sA}\sin(4A) - \frac{4}{s^2}e^{-sA}\cos(4A) + \frac{1}{s}(1)(0) + \frac{4}{s^2}(1)(1)\right]$$

$$= \frac{s^2}{s^2+16}\left[\frac{-1}{s}e^{-sA}\sin(4A) - \frac{4}{s^2}e^{-sA}\cos(4A) + \frac{4}{s^2}\right]$$

Take lim as $A\to\infty$. When $s>0$, get:

$$\mathcal{L}\{\sin(4t)\} = \frac{s^2}{s^2+16}\left(\frac{4}{s^2}\right) = \boxed{\frac{4}{s^2+16}}, \quad s>0$$

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Part II: *Complete 4 of the 5 problems. Clearly mark the one problem you wish to omit.*

5. Consider the initial value problem

$$\begin{cases} (9x^2 + y - 1) - (4y - x)y' = 0 \\ y(1) = 0 \end{cases}$$

3 *a.)* Show that the differential equation is exact.

$M(x,y) = 9x^2 + y - 1 \qquad N(x,y) = x - 4y$

$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned}\right\}$ Therefore Exact!

7 *b.)* Solve the initial value problem.

$\psi = \int M\,dx = \int 9x^2 + y - 1\,dx = 3x^3 + xy - x + h(y)$

$\psi = \int N\,dy = \int x - 4y\,dy = xy - 2y^2 + g(x)$

Thus, the soln is:

$\psi(x,y) = \boxed{3x^3 + xy - x - 2y^2 = C}$

at $(x,y) = (1,0)$: $3 + 0 - 1 - 0 = 2 = C$

Thus, the soln of the IVP is:

$\boxed{3x^3 + xy - x - 2y^2 = 2}$

5–6 pts

Solving for y:

$\underbrace{-2y^2}_{} + \underbrace{xy}_{B} + \underbrace{3x^3 - x - 2}_{C} = 0$

$y = \dfrac{-x \pm \sqrt{x^2 + 8(3x^3 - x - 2)}}{-4}$

$y = \dfrac{x}{4} \pm \dfrac{\sqrt{x^2 + 24x^3 - 8x - 16}}{4}$

when $x = 1$, $x^2 + 24x^3 - 8x - 16 = 1$,

so the soln is:

$\boxed{y = \dfrac{x}{4} - \dfrac{\sqrt{24x^3 + x^2 - 8x - 16}}{4}}$

10 pts

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6. Consider the initial value problem

$$\begin{cases} y' + \dfrac{4t}{y} = 0 \\ y(0) = y_0 \end{cases}$$

4 *a.)* Use the Fundamental Existence and Uniqueness Theorem (FEUT) to determine for what values of y_0 a solution to the IVP does *not* exist.

$y' = f(t,y) = -4t/y$

$\frac{\partial f}{\partial y} = \frac{4t}{y^2}$

These are both defined for $y_0 \neq 0$.

So $\boxed{y_0 = 0}$ is the only initial condition where a soln does not exist.

!

$y_0 > 0$ 6 *b.)* Assuming that ~~the IVP has a solution at the given initial point~~ ?, find the solution.

$\frac{dy}{dt} = \frac{-4t}{y} \Rightarrow y\,dy = -4t\,dt$

$\frac{1}{2}y^2 = -2t^2 + C$

$y^2 = -4t^2 + C$

$y = \pm\sqrt{C - 4t^2}$ ← + since $y_0 > 0$

$y(0) = y_0 = \pm\sqrt{C} \Rightarrow C = y_0^2$

The solution is thus:

$$\boxed{y(t) = \sqrt{y_0^2 - 4t^2}}$$

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7. Determine the maximum interval on which the solutions of the IVP are guaranteed to exist *without solving the IVP.*

5 a.) $(t-3)y' + (\ln t)y = 2t; \quad y(1) = 2$

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

$p(t) = \frac{\ln t}{t-3}$, dom $(p) = (0,3) \cup (3, \infty)$

$g(t) = \frac{2t}{t-3}$, dom$(g) = t \neq 3$.

since $t_0 = 1$, max. interval guaranteed is $\boxed{(0,3)}$

! 5 b.) $(4-t^2)y' + y = \tan t; \quad y(\overset{3}{2}) = 3$

$$y' + \frac{1}{4-t^2}y = \frac{\tan t}{4-t^2}$$

$p(t) = \frac{1}{4-t^2}$, dom $(p) = t \neq \pm 2$

$g(t) = \frac{\tan t}{4-t^2}$, dom$(g) = \begin{cases} t \neq \frac{\pi}{2} \pm \pi n \\ t \neq \pm 2 \end{cases}$

$t_0 = 3$, so interval is $\boxed{(2, \frac{3\pi}{2})}$

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8. Consider the differential equation

$$t^2y'' - 4ty' + 6y = 0; \quad t > 0.$$

7 *a.)* Given that $y_1(t) = t^2$ is a solution, use the method of reduction of order to find another solution y_2. Clearly identify y_2.

Let: $y = t^2 v$

$y' = 2tv + t^2 v'$

$y'' = 2v + 2tv' + 2tv' + t^2 v''$

$= 2v + 4tv' + t^2 v''$

Plugging in:

$\cancel{2t^2 v} + \cancel{4t^3 v'} + t^4 v'' - \cancel{8t^2 v} - \cancel{4t^3 v'} + \cancel{6t^2 v} = 0$

$t^4 v'' = 0, \; t \neq 0$, so

$v'' = 0$

then $v = C_1 + C_2 t$

so $y = \underset{\uparrow \text{ old}}{C_1 t^2} + \underset{\uparrow \text{ new}}{C_2 t^3}$ thus $\boxed{y_2 = t^3}$

3 *b.)* Find the Wronskian $W(y_1, y_2)$.

$$W(t^2, t^3) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = \boxed{t^4}$$

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9. Consider the initial value problem

$$\begin{cases} y' = \frac{t}{y} \\ y(0) = 1 \end{cases}$$

2 *a.)* Write the Picard operator $\mathcal{P}f$ for the initial value problem.

$$\boxed{\mathcal{P}f(\varphi) = \int_0^t \frac{s}{\varphi}\,ds + 1}$$

4/4 *b.)* Letting $\varphi_0 = 1$, use Picard's iterative method to calculate φ_1 and φ_2. Clearly identify each one.

$\varphi_0 = 1$

$$\varphi_1 = \mathcal{P}f(\varphi_0) = \int_0^t \frac{s}{1}\,ds + 1 = \frac{1}{2}s^2\Big|_0^t + 1 = \boxed{\frac{1}{2}t^2 + 1 = \varphi_1}$$

$$\varphi_2 = \mathcal{P}f(\varphi_1) = \int_0^t \frac{s}{\frac{1}{2}s^2+1}\,ds + 1 = \int_0^t \frac{2s}{s^2+2}\,ds + 1 = \ln(s^2+2)\Big|_0^t + 1$$

$$= \boxed{\ln(t^2+2) - \ln 2 + 1 = \varphi_2}$$