Math 555: Differential Equations I
Midterm Exam # 4

Friday, 22 November 2013

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points. Show all work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I: Complete all 3 problems.

1. Use the definition of the Laplace transform to find the Laplace transform of

$$f(t) = te^{2t}.$$

Be sure to show all work, and treat the improper integral(s) correctly.

$$\mathcal{L}\{f(t)\} = \int_0^\infty te^{2t}e^{-st}\,dt = \int_0^\infty t\,e^{-(s-2)t}\,dt$$

$$= \lim_{A\to\infty}\int_0^A t\,e^{-(s-2)t}\,dt = \lim_{A\to\infty}\int_0^A t\,\frac{d}{dt}\left[\frac{-1}{s-2}e^{-(s-2)t}\right]dt$$

$$= \lim_{A\to\infty}\left[\frac{-t}{s-2}e^{-(s-2)t}\Big|_0^A + \frac{1}{s-2}\int_0^A e^{-(s-2)t}\,dt\right]$$

$$= \lim_{A\to\infty}\left[\frac{-A}{s-2}e^{-(s-2)A} - \frac{1}{(s-2)^2}e^{-(s-2)t}\Big|_0^A\right]$$

$$= \lim_{A\to\infty}\left[\frac{-A}{s-2}e^{-(s-2)A} - \frac{1}{(s-2)^2}e^{-(s-2)A} + \frac{1}{(s-2)^2}\right]$$

$$= \frac{1}{(s-2)^2} \quad \text{for } s>2.$$

Therefore, $\boxed{F(s) = \frac{1}{(s-2)^2}\,,\ s>2}$

2. Use the definition of the Laplace transform to find the Laplace transform of

$$f(t) = u_5(t)(t-1)^2.$$

Be sure to show all work, and treat the improper integral(s) correctly.

$$\mathcal{L}\{f(t)\} = \int_0^\infty u_5(t)\,(t-1)^2 e^{-st}\,dt = \int_5^\infty (t-1)^2 e^{-st}\,dt$$

$$= \lim_{A\to\infty} \int_5^A (t-1)^2 e^{-st}\,dt = \lim_{A\to\infty} \int_5^A (t-1)^2 \left[\frac{d}{dt}\left(\frac{-1}{s}e^{-st}\right)\right] dt$$

$$= \lim_{A\to\infty}\left[-\frac{(t-1)^2}{s} e^{-st}\Big|_5^A + \frac{1}{s}\int_5^A 2(t-1)\frac{d}{dt}\left[\frac{-1}{s}e^{-st}\right] dt\right]$$

$$= \lim_{A\to\infty}\left[-\frac{(t-1)^2}{s} e^{-st}\Big|_5^A - \frac{2(t-1)}{s^2} e^{-st}\Big|_5^A + \frac{2}{s^2}\int_5^A e^{-st}\,dt\right]$$

$$= \lim_{A\to\infty}\left[\underbrace{\cancel{\frac{-(A-1)^2}{s} e^{-sA}}}_{\to 0} + \frac{(5-1)^2}{s} e^{-5s} - \underbrace{\cancel{\frac{2(A-1)}{s^2} e^{-sA}}}_{\to 0} + \frac{2(5-1)}{s^2} e^{-5s} - \underbrace{\cancel{\frac{2}{s^3} e^{-sA}}}_{\to 0} + \frac{2}{s^3} e^{-5s}\right]$$

when $s > 0$, this becomes

$$= \boxed{e^{-5s}\left[\frac{16}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right] = F(s)}$$

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3. Find $\mathcal{L}\{f''(t)\}$, where $f(t) = \cancel{t}e^{3t}\cos(t)$.

You may use the table for help, if necessary. Write as a single fraction.

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$f(0) = (1)(1) = 1$$

$$\cancel{f'(t) = (te^{3t})(-\sin t) + (te^{3t})'\cos t}$$

$$\cancel{= -te^{3t}\sin t + (3te^{3t} + e^{3t})\cos t}$$

$$f'(t) = 3e^{3t}\cos t - e^{3t}\sin t$$

$$f'(0) = 3(1)(1) - 1(0) = 3$$

$$\cancel{f'(0) = 0 + (0+1)(1) = 1}$$

$$\mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2+1} = \frac{s-3}{s^2-6s+10}$$

so,

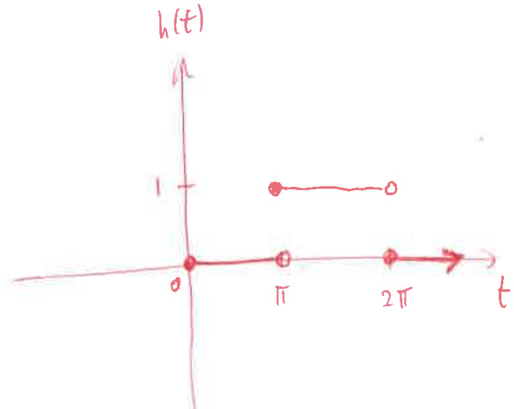
$$\mathcal{L}\{f''(t)\} = s^2\left(\frac{s-3}{s^2-6s+10}\right) - s - 3 = \frac{\cancel{s^3} - \cancel{3s^2} - \cancel{s^3} + \cancel{6s^2} - 10s - \cancel{3s^2} + 18s - 30}{s^2-6s+10}$$

$$= \boxed{\frac{8s-30}{s^2-6s+10} = \mathcal{L}\{f''(t)\}}$$

Part II: Complete 2 of the 3 problems. Clearly mark the one problem you wish to omit.

4. (*a.*) Write the piecewise function $h(t)$ in terms of step functions $u_c(t)$, and sketch the graph of $y = h(t)$.

$$h(t) = \begin{cases} 0 & 0 \le t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & 2\pi \le t. \end{cases}$$



$h(t) = u_\pi(t) - u_{2\pi}(t)$

(b) Find $H(s) = \mathcal{L}\{h(t)\} = \dfrac{e^{-\pi s}}{s} - \dfrac{e^{-2\pi s}}{s}$.

(c.) Find the inverse Laplace Transform of

$$F(s) = \frac{2s+1}{s^2 - 2s + 2}.$$

$$F(s) = \frac{2s+1}{s^2-2s+1\ \ +1} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1^2} + \frac{3(1)}{(s-1)^2+1^2}$$

so, $\boxed{2e^{+t}\cos t + 3e^{+t}\sin t = f(t)}$

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5. Prove the theorem:

If $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > a \geq 0$, *and if* $c > 0$ *is a constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s).$$

Show all of your work.

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt = \int_c^\infty f(t-c)\,e^{-st}\,dt$$

let $u = t-c$ so $t = u+c$ and $dt = du$

when $t=c$, $u=0$, so the integral becomes

$$\int_0^\infty f(u)\,e^{-s(u+c)}\,du = \int_0^\infty f(u)\,e^{-su}\,e^{-cs}\,du$$

$$= e^{-cs}\int_0^\infty f(u)\,e^{-su}\,du$$

$$= e^{-cs}\,\mathcal{L}\{f(t)\}.$$

$$= e^{-cs}F(s)$$

$\square$

6. Use Laplace transforms to solve the initial value problem:

$$y^{(4)} - 4y = 0; \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = 0.$$

$$\mathcal{L}\{y^{(4)}\} = s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$$

$$= s^4 Y - s^3 - 0 + 2s - 0$$

The DE becomes

$$s^4 Y - s^3 + 2s - 4Y = 0$$

$$Y(s^4 - 4) = s^3 - 2s = s\cancel{(s^2-2)}$$

$$Y = \frac{s^3 - 2s}{s^4 - 4} = \frac{s^3 - 2s}{(s^2+2)\cancel{(s^2-2)}} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2-2}$$

Then $s^3 - 2s = (As+B)(s^2-2) + (Cs+D)(s^2+2)$

$$= As^3 + Bs^2 - 2As - 2B + Cs^3 + Ds^2 + 2Cs + 2D$$

s^3: $1 = A + C$

s^2: $0 = B + D$

s: $-2 = -2A + 2C$

#: $0 = -2B + 2D$

$$\left.\begin{array}{l} B = D = 0 \\ A = 1 \\ C = 0 \end{array}\right\} \text{duh,}$$

so

$$Y = \frac{s}{s^2+2}$$

$1 = A + C$

$1 = A - C$

$2 = 2A$

Therefore,

$$\boxed{y(t) = \cos\sqrt{2}t}$$