Math 555: Differential Equations I
Midterm Exam # 3

Due: Monday, 4 November 2013

Late submissions will not be accepted!

Name: Key

Instructions: Complete all 5 problems, showing enough work. Each completed problem is worth 20 points. Follow the instructions carefully. Write your name on each page.

Good Luck!

Name:______________________________

1. Find the general solution of the following problem.

$$y'' - 2y' + y = \frac{2e^t}{1+t^2}$$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1$

$y_1(t) = e^t$

$y_2(t) = te^t$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} = (1+t)e^{2t} - te^{2t} = e^{2t}$$

By variation of parameters, the general solution is:

$$y(t) = \underbrace{y_1(t)\int_{t_0}^{t} \frac{-y_2(s)\,g(s)}{W(s)}\,ds}_{I_1} + \underbrace{y_2(t)\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}\,ds}_{I_2}$$

$$I_1 = \int_{t_0}^{t} \frac{-s\cancel{e^s}\left(2\frac{\cancel{e^s}}{1+s^2}\right)}{\cancel{e^{2s}}}\,ds = \int_{t_0}^{t} \frac{-2s}{1+s^2}\,ds \qquad u = 1+s^2, \quad du = 2s\,ds$$

$$= \int_{u_0}^{1+t^2} \frac{-1}{u}\,du = -\ln(1+t^2) + C_1$$

$$I_2 = \int_{t_0}^{t} \frac{2\cancel{e^{2s}}}{\cancel{e^{2s}}(1+s^2)}\,ds = 2\arctan(s)\Big|_{t_0}^{t} = 2\arctan(t) + C_2$$

General Sol'n is thus:

$$\boxed{y = C_1e^t + C_2te^t - e^t\ln(1+t^2) + 2te^t\arctan(t)}$$

2. (*a.*) Find a power series representation for the function $f(x) = \ln(x)$ around the point $x_0 = 1$.

	at $x_0=1$:
$f(x) = \ln x$	0
$f'(x) = \frac{1}{x}$	1
$f''(x) = \frac{-1}{x^2}$	-1
$f'''(x) = \frac{2}{x^3}$	2
$f^{(4)}(x) = \frac{-2\cdot 3}{x^4}$	$-2\cdot 3$
$f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5}$	$2\cdot 3\cdot 4$
$\vdots$	
$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$	$(-1)^{n+1}(n-1)!$

So the Taylor Series is:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{(n-1)!}}{\cancel{n!}}(x-1)^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$$

(*b.*) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{5^n}(x-3)^n.$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right| = \frac{1}{5}\left|\frac{n+1}{n}\right| = \frac{1}{5}\left|1+\frac{1}{n}\right| \to \frac{1}{5} \quad \text{as } n \to \infty$$

Thus, by the ratio test, converges when

$$\frac{1}{5}|x-3| < 1$$

or $|x-3| < 5$.

$$\therefore \boxed{R=5}$$

Name:________________________

3. Find power series solutions to the DE

$$y'' + x^2 y = 0.$$

(*a*.) Derive and clearly label the recurrence relation.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$y'' + x^2 y = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x^2 \sum_{n\geq 0} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+2} =$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n\geq 2}^{\infty} a_{n-2}x^n = 2a_2 + 2\cdot 3 a_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-2}\right]x^n$$

$\Rightarrow$ $\boxed{\begin{matrix} a_2 = 0 \\ a_3 = 0 \end{matrix}}$ and $(n+2)(n+1)a_{n+2} = -a_{n-2}$ or $\boxed{a_{n+2} = \frac{-1}{(n+2)(n+1)} a_{n-2},\ n \geq 2}$ rec. rel. !

corr. $n=0$, $n=1$

(*b*.) Write the first 4 terms for each of the fundamental solutions y_1 and y_2.

MORE ROOM!

$n=2$: $a_4 = \frac{-1}{3\cdot 4} a_0$

$n=6$: $a_8 = \frac{-1}{7\cdot 8} a_4 = \frac{1}{3\cdot4\cdot7\cdot8} a_0$

$n=10$: $a_{12} = \frac{-1}{11\cdot 12} a_8 = \frac{-1}{3\cdot4\cdot7\cdot8\cdot11\cdot12} a_0$

similarly,

$n=3$: $a_5 = \frac{-1}{4\cdot 5} a_1$

$a_9 = \frac{+1}{4\cdot5\cdot8\cdot9} a_1$

$a_{13} = \frac{-1}{4\cdot5\cdot8\cdot9\cdot12\cdot13} a_1$

$a_0 y_1 = a_0 - a_0 \frac{1}{3\cdot4}x^4 + a_0 \frac{1}{3\cdot4\cdot7\cdot8}x^8 - a_0 \frac{1}{3\cdot4\cdot7\cdot8\cdot11\cdot12}x^{12}$

so $\boxed{y_1 = 1 - \frac{1}{3\cdot4}x^4 + \frac{1}{3\cdot4\cdot7\cdot8}x^8 - \frac{1}{3\cdot4\cdot7\cdot8\cdot11\cdot12}x^{12}}$

and

$\boxed{y_2 = x - \frac{1}{4\cdot5}x^5 + \frac{1}{4\cdot5\cdot8\cdot9}x^9 - \frac{1}{4\cdot5\cdot8\cdot9\cdot12\cdot13}x^{13}}$

~~(*c*.)~~ Write the fundamental solutions in Sigma-notation, and write and clearly label the general solution y.

uh, no.

4. (*a.*) Determine a lower bound for the radius of convergence of series solutions to the DE centered around the point x_0.

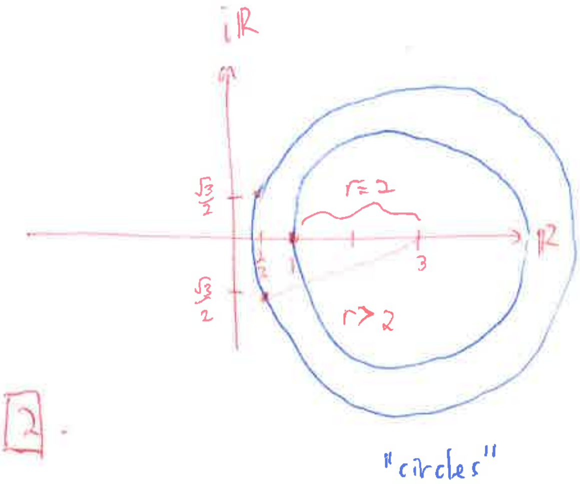
$$(1+x^3)y'' + 4xy' + y = 0\,; \quad x_0 = 3$$

$x^3+1 = (x+1)(x^2 - x + 1) = 0$

$x = -1$ $\quad (x^2 - x + \frac{1}{4}) + \frac{3}{4} = 0$

$(x - \frac{1}{2})^2 = -3/4$

$x = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$



So lower bound for R is $\boxed{2}$.

(*b.*) Find a power series representation for $f(x) = \dfrac{-6x^2}{(1+x^3)^2}$ around the point $x_0 = 0$.

$\int f(x)\,dx = \int \frac{-6x^2}{(1+x^3)^2}\,dx \quad \left\{ \begin{array}{l} u = 1+x^3 \\ du = 3x^2\,dx \end{array} \right\} = \int \frac{-2}{u^2}\,du = -2\int u^{-2}\,du = 2u^{-1}$

$= \frac{2}{1+x^3}$

$\frac{2}{1+x^3}$ is geometric w/ $a = 2$ and $r = -x^3$

so $\int f(x)\,dx = \sum_{n=0}^{\infty} 2(-x^3)^n = 2\sum_{n=0}^{\infty} (-1)^n x^{3n}$

$f(x) = \left(\int f(x)\,dx\right)' = 2\left(\sum_{n=0}^{\infty} (-1)^n x^{3n}\right)' = 2\sum_{n=1}^{\infty} (-1)^n 3n\, x^{3n-1}$

or

$$\boxed{f(x) = \sum_{n=1}^{\infty} 6(-1)^n n\, x^{3n-1}}$$

Name:________________________

5. (*a.*) Determine the general solution of the DE that is valid in any interval not including the singular point. Identify the singular point.

$$(x-2)^2y'' + 5(x-2)y' + 8y = 0$$

Singular pt: $x=2$:

let $u = x-2$: $\quad u^2 y'' + 5u y' + 8y = 0$

$$r^2 + 4r + 8 = 0$$

$$(r+2)^2 = -4$$

$$r = -2 \pm 2i, \quad \lambda = -2, \ \mu = 2$$

soln: $y = (u)^{-2}\left(c_1 \cos(2\ln|u|) + c_2 \sin(2\ln|u|)\right)$

$$\boxed{y = \frac{1}{(x-2)^2}\left(c_1 \cos\left(\ln(x-2)^2\right) + c_2 \sin\left(\ln(x-2)^2\right)\right)}$$

(*b.*) Determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point.

$$y'' - 2xy' + \lambda y = 0\,; \quad \lambda \neq 0.$$

$P(x) = 1$ so $P(1/\xi) = 1$

$Q(x) = -2x$ so $Q(1/\xi) = -2/\xi$

$R(x) = \lambda$ so $R(1/\xi) = \lambda$

$$A = \frac{1}{P(1/\xi)}\left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right] = \frac{2}{\xi} - \frac{2}{\xi^3} = \frac{2\xi^2 - 2}{\xi^3}$$

$$B = \frac{R(1/\xi)}{\xi^4 P(1/\xi)} = \frac{\lambda}{\xi^4}$$

$\xi A = \dfrac{2\xi^2 - 2}{\xi^2}$, $\quad \xi^2 B = \dfrac{\lambda}{\xi^2}$. Neither is defined for $\xi = 0$, so the pt at ∞ is $\boxed{\text{irregular}}$.