Math 555: Differential Equations I
Midterm Exam # 2

Friday, 11 October 2013

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points. Show all work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

Name: Key

Part I: Complete all 3 problems.

1. (*a*.) Solve the homogeneous DE:

$$y'' - 8y' + 12y = 0$$

$r^2 - 8r + 12 = 0$

$(r-6)(r-2) = 0$

$r = 6, 2$

so $\boxed{y = C_1 e^{6t} + C_2 e^{2t}}$

(*b*.) Solve the inhomogeneous DE:

$$y'' - 8y' + 12y = 24t^2 - 32t + 28$$

Guess: $Y(t) = At^2 + Bt + C$

$Y' = 2At + B$

$Y'' = 2A$

$Y'' - 8Y' + 12Y = 2A - 16At - 8B + 12At^2 + 12Bt + 12C = 24t^2 - 32t + 28$

so:

$$\left.\begin{aligned} 12A &= 24 \\ 12B - 16A &= -32 \\ 2A - 8B + 12C &= 28 \end{aligned}\right\} \quad \begin{aligned} &A = 2 \\ &12B - 32 = -32 \Rightarrow B = 0 \\ &4 - 0 + 12C = 28 \Rightarrow C = 2 \end{aligned}$$

so $Y(t) = 2t^2 + 2$

The general sol'n is:

$$\boxed{y = 2t^2 + 2 + C_1 e^{6t} + C_2 e^{2t}}$$

2. Solve the initial value problem:

$$y'' - 2y' + 5y = 8\sin(t) - 4\cos(t)\,; \quad y(0) = 3, \quad y'(0) = 9.$$

homog: $r^2 - 2r + 5 = 0$

$$r^2 - 2r + 1 = -4$$
$$(r-1)^2 = -4$$
$$r - 1 = \pm 2i$$
$$r = 1 \pm 2i$$
$$\boxed{y = e^{t}\left(c_1\cos(2t) + c_2\sin(2t)\right)}$$

inhomog.:

guess: $Y = A\cos t + B\sin t$

$$Y' = -A\sin t + B\cos t$$
$$Y'' = -A\cos t - B\sin t$$

$$Y'' - 2Y' + 5Y = -A\cos t - B\sin t - 2(-A\sin t + B\cos t) + 5(A\cos t + B\sin t)$$
$$= \sin t\,(-B + 2A + 5B) + \cos t\,(-A - 2B + 5A)$$
$$= \sin t\,(2A + 4B) + \cos t\,(4A - 2B) = 8\sin t - 4\cos t$$

$$2A + 4B = 8$$
$$4A - 2B = -4$$

$$\left.\begin{array}{l}\det A = -20\\ \det A_1 = 0\\ \det A_2 = -40\end{array}\right\}\text{ so } A = 0, \quad B = \frac{-40}{-20} = 2$$

so $Y = 2\sin t$ and $y = e^{t}\left(c_1\cos(2t) + c_2\sin(2t)\right) + 2\sin t$

$$y' = c_1\left(e^{t}\cos(2t) - 2e^{t}\sin(2t)\right) + c_2\left(e^{t}\sin(2t) + 2e^{t}\cos(2t)\right) + 2\cos t$$

$y(0) = \boxed{c_1 = 3}$

$y'(0) = 3 + 2c_2 + 2 = 9$

$$2c_2 = 4$$
$$\boxed{c_2 = 2}$$

so finally,

$$\boxed{y = 3e^{t}\cos(2t) + 2e^{t}\sin(2t) + 2\sin(t)}$$

Name: Key

3. Find the general solution:

$$y'' + 18y' + 81y = 6e^{-9t}.$$

homog: $r^2 + 18r + 81 = 0$

$(r+9)^2 = 0$

$r = -9$

$y = c_1 e^{-9t} + c_2 t e^{-9t}$

inhomog.: Guess $Y = At^2 e^{-9t}$

$Y' = 2Ate^{-9t} - 9At^2 e^{-9t}$

$Y'' = 2Ae^{-9t} - 18Ate^{-9t} - 18Ate^{-9t} + 81At^2 e^{-9t}$

$Y'' + 18Y' + 81Y = 2Ae^{-9t} - \cancel{36Ate^{-9t}} + \cancel{81At^2 e^{-9t}} + 18(\cancel{2Ate^{-9t}} - \cancel{9At^2 e^{-9t}}) + \cancel{81At^2 e^{-9t}}$

$= 2Ae^{-9t} = 6e^{-9t}$

$\Rightarrow A = 3$ and $Y(t) = 3t^2 e^{-9t}$

The general sol'n is:

$$\boxed{y = c_1 e^{-9t} + c_2 t e^{-9t} + 3t^2 e^{-9t}}$$

Part II: Complete 2 of the 3 problems. Clearly mark the one problem you wish to omit.

4. (*a.*) Use Abel's theorem to calculate the Wronksian for Legendre's Equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

$u = 1-x^2$
$du = -2x\,dx$

$$W = e^{-\int \frac{-2x}{1-x^2}dx} = e^{-\ln|1-x^2|+C} = \boxed{\frac{C}{1-x^2}}$$

(*b.*) Does the initial value problem

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0\,; \quad y(1) = 3, \quad y'(1) = -2$$

have a solution? Why or why not? (Do *not* try to solve this IVP.)

No. because $x=1$ is not in the domain of $p(x) = \frac{-2x}{1-x^2}$. Also, the Wronskian is undefined for $x=1$.

Name: Key

5. (*a*.) Write the Picard operator $\mathcal{P}f$ for the initial value problem

$$y' = ty + t^2\,; \quad y(0) = 1$$

$$\boxed{\mathcal{P}f(y) = \int_0^t sy + s^2\,ds + 1}$$

(*b*.) Letting $\varphi_0 = \overset{1}{\cancel{y_0}}$, use Picard's iterative method to calculate φ_1, φ_2, and φ_3. Clearly mark each one.

$$\varphi_1 = \mathcal{P}f(\varphi_0) = \int_0^t s + s^2\,ds + 1 = \frac{1}{2}s^2 + \frac{1}{3}s^3 \Big|_0^t + 1 = \boxed{\frac{1}{2}t^2 + \frac{1}{3}t^3 + 1}$$

$$\varphi_2 = \mathcal{P}f(\varphi_1) = \int_0^t s\left(\frac{1}{2}s^2 + \frac{1}{3}s^3 + 1\right) + s^2\,ds + 1$$

$$= \int_0^t \frac{1}{2}s^3 + \frac{1}{3}s^4 + s + s^2\,ds + 1$$

$$= \frac{1}{8}s^4 + \frac{1}{15}s^5 + \frac{1}{2}s^2 + \frac{1}{3}s^3 \Big|_0^t + 1 = \boxed{\frac{1}{8}t^4 + \frac{1}{15}t^5 + \frac{1}{2}t^2 + \frac{1}{3}t^3 + 1}$$

$$\varphi_3 = \mathcal{P}f(\varphi_2) = \int_0^t s\left(\frac{1}{8}s^4 + \frac{1}{15}s^5 + \frac{1}{2}s^2 + \frac{1}{3}s^3 + 1\right) + s^2\,ds + 1$$

$$= \int_0^t \frac{1}{8}s^5 + \frac{1}{15}s^6 + \frac{1}{2}s^3 + \frac{1}{3}s^4 + s + s^2\,ds + 1$$

$$= \boxed{\frac{1}{105}t^7 + \frac{1}{48}t^6 + \frac{1}{15}t^5 + \frac{1}{8}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 + 1}$$

6. Consider the second order linear DE

$$(x-1)y'' - xy' + y = 0\,; \quad x > 1$$

$(a.)$ Given that $y_1 = e^x$ is a solution, use the method of reduction of order to find another solution y_2. Clearly label y_2.

let $y = v(x)e^x$

$y' = ve^x + v'e^x$

$y'' = ve^x + 2v'e^x + v''e^x$

$(x-1)y'' - xy' + y = x\cancel{e^x}(v'' + 2v' + \cancel{v}) - \cancel{e^x}(v'' + 2v' + \cancel{v}) - x\cancel{e^x}(v' + \cancel{v}) + \cancel{e^x}\cancel{v} = 0$

$\Rightarrow \quad (x-1)v'' + (x-2)v' = 0$

$u = x-1$

$v'' = \dfrac{(2-x)v'}{x-1} \qquad \dfrac{x}{x-1} = \dfrac{u+1}{u} = 1 + \dfrac{1}{u} = 1 + \dfrac{1}{x-1}$

$\dfrac{dv'}{v'} = \left(\dfrac{2}{x-1} - \dfrac{x}{x-1}\right)dx$

$\ln(v') = \int \dfrac{1}{x-1} - 1\, dx = \ln(x-1) - x + C_1$

$v' = C_1e^{-x}(x-1) = C_1xe^{-x} - C_1e^{-x}$

~~$(b.)$ Calculate the Wronskian $W(y_1, y_2)$.~~

$v = \int C_1xe^{-x}dx - \int C_1e^{-x}dx$

$v = C_1\left(-xe^{-x} - e^{-x}\right) + C_1e^{-x} + C_2$

$v = C_1xe^{-x} + C_2$

So $y = ve^x = C_1x + C_2e^x$

Therefore $\boxed{y_2 = x}$