Math 555: Differential Equations I
Midterm Exam # 1

Friday, 13 Sep 2013

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points. Show all work, and follow the instructions carefully. Write your name on each page. You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

Name:____________________________

Part I: Complete all 3 problems.

1. Find the particular solution of the IVP.

$$y' = \frac{2xy}{1+x^2} \; ; \qquad y(0) = 3$$

$$\frac{dy}{dx} = \frac{2xy}{1+x^2}$$

$$\frac{dy}{y} = \frac{2x}{1+x^2}\,dx$$

$$\ln(y) = \ln(1+x^2) + C$$

$$y = C(1+x^2)$$

$$y(0) = 3 \Rightarrow y(0) = 3 = C(1+0^2) = C$$

So $\boxed{y = 3(1+x^2)}$

Name:______________________________

2. Find the general solution of the DE.

$$y' + 2ty = 4t^3$$

$p(t) = 2t \Rightarrow \mu(t) = e^{\int 2t\,dt} = e^{t^2}$

$$y = e^{-t^2}\int_{t_0}^{t} e^{s^2}(4s^3)\,ds + Ce^{-t^2}$$

To integrate: let $x = s^2$ $\quad dx = 2s\,ds$

$$\int 4s^3 e^{s^2}\,ds = \int 2s^2 e^{s^2}\, 2s\,ds = 2\int x e^x\,dx$$

$u = x \quad dv = e^x\,dx$

$du = dx \quad v = e^x$

$\Rightarrow 2\int x e^x\,dx = 2xe^x - 2e^x$

$$= 2s^2 e^{s^2} - 2e^{s^2}\Big|_{t_0}^{t}$$

Plugging back into y yields:

$$y = e^{-t^2}\left(2t^2 e^{t^2} - 2e^{t^2}\right) + Ce^{-t^2}$$

or

$$\boxed{y = 2t^2 - 2 + Ce^{-t^2}}$$

Name:______________________________

3. Find the general solution of the DE.

$$\left(6(x+y)^2 + y^2 e^{xy} + 12x^3\right) + \left(e^{xy} + xye^{xy} + \cos y + 6(x+y)^2\right)y' = 0$$

$$\left.\begin{aligned} \partial_y M &= 12(x+y) + 2ye^{xy} + xy^2e^{xy} \\ \partial_x N &= ye^{xy} + ye^{xy} + x^2ye^{xy} + 12(x+y) \end{aligned}\right\} = \text{exact.}$$

$$\psi(x,y) = \int 6(x+y)^2 + y^2e^{xy} + 12x^3 \, dx = 2(x+y)^3 + ye^{xy} + 3x^4 + f(y)$$

$$\psi(x,y) = \int e^{xy} + xye^{xy} + \cos y + 6(x+y)^2 \, dy = \frac{1}{x}e^{xy} + ye^{xy} - \frac{1}{x}e^{xy}$$

$$+ \sin y + 2(x+y)^3 + h(x)$$

So,

$$\psi(x,y) = \boxed{2(x+y)^3 + ye^{xy} + 3x^4 + \sin y = C}$$

Name:______________________________

Part II: Complete 2 of the 3 problems. Clearly mark the one problem you wish to omit.

4. *a.)* Find all values of r such that $y = e^{rt}$ is a solution of the DE.

$$y'' + 12y' - 28y = 0$$

$$\left.\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2e^{rt} \end{aligned}\right\}$$

$r^2e^{rt} + 12re^{rt} - 28e^{rt} = 0$

$e^{rt}(r^2+12r-28) = 0$

$(r+14)(r-2) = 0$

$\boxed{r = 2, -14}$

b.) Verify that $\varphi(t) = 3t + t^2$ is a solution of the DE.

$$ty' - y = t^2$$

$\varphi'(t) = 3+2t$

Plug in to LHS:

$t(3+2t) - (3t+t^2) = 3t + 2t^2 - 3t - t^2$

$= 0.$ $\square$

Name:________________________________

5. *a.*) Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

$$\left.\begin{aligned}\partial_y M(x) &= 0\\ \partial_x N(y) &= 0\end{aligned}\right\}$$ These are equal, so the DE is exact!

b.) Solve the IVP.

$$(2x - y) + (2y - x)y' = 0\ ; \quad y(1) = 3$$

$$\left.\begin{aligned}\partial_y M &= -1\\ \partial_x N &= -1\end{aligned}\right\} =, \text{ so exact.}$$

$\int M\,dx = \int 2x-y\,dx = x^2 - xy + f(y)$

$\int N\,dy = \int 2y-x\,dy = y^2 - xy + h(x)$

$\boxed{y = \frac{1}{2}\left(x + \sqrt{28-3x^2}\right)}$

so $\psi(x,y) = x^2 + y^2 - xy = C$

Plugging in $x=1, y=3$: $1^2 + 3^2 - 1(3) = 1 + 9 - 3 = 7$

to solve for y:

$y^2 - xy + x^2 - 7 = 0$

So soln is:

$\boxed{x^2 + y^2 - xy = 7}$

$y = \dfrac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2}$

$y = \frac{x}{2} \pm \frac{1}{2}\sqrt{28-3x^2}$

need + for I.C.: $\boxed{y = \frac{1}{2}\left(x + \sqrt{28-3x^2}\right)}$

Name:______________________________

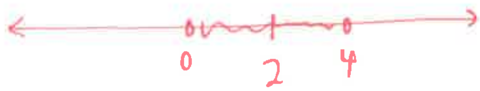
6. Determine the maximum interval on which the solutions of the DE are guaranteed to exist *without solving the DE.*

a.)

$$t(t-4)y' + y = 0\ ; \quad y(2) = 1$$

$$y' + \frac{1}{t(t-4)}\, y = 0$$

$t \neq 0, 4$



domain of sol'n is: $(0,4)$ or $\boxed{0 < t < 4}$.

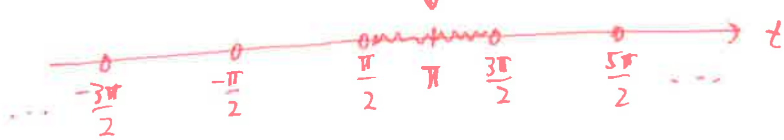
b.)

$$y' + (\tan t)y = \sin t\ ; \quad y(\pi) = 0$$

dom(tan): $t \neq \pm n\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$

dom(sin): $\mathbb{R}$

$\downarrow$



so domain of the sol'n is

$$\boxed{\frac{\pi}{2} < t < \frac{3\pi}{2}}$$