Math 555: Differential Equations
Numerical Methods Project

Due: Wednesday, 23 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name:____

Instructions: Complete each problem on your own paper. Print all spreadsheets and graphs.

Please use Excel (or your favorite alternative) when appropriate. I know that some of you may know “better" ways, but I want this project done using spreadsheets.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct “answer” is not good enough; I need to see how you got it!

Good Luck!

1. [30] Consider the initial value problem

$$\begin{cases} y' = 2y - e^{2t}; \\ y(0) = 1 \end{cases}$$

a.) Find the exact solution analytically.

b.) Use Euler's method, Improved Euler's method, and the Runge-Kutta method with a step-size of $h = 0.1$ to find approximate values of the solution on the interval $[0,2]$. Plot these together with the exact solution on the same set of axes.

c.) Repeat part *b.* with a step-size of $h = 0.05$.

d.) Calculate the error in each of the three approximations that you found in part *c.* by subtracting the appropriate values from the values of the actual solution at each t-value. Make a dot plot of these error terms.

2. [30] Consider the initial value problem

$$\begin{cases} y' = \dfrac{3t^2}{3y^2 - 4}; \\ y(0) = 0 \end{cases}$$

a.) Use the FEUT to show that a solution of this differential equation exists for this initial data.

b.) Use the Runge-Kutta method with various step-sizes to estimate how far the solution can be extended to the right. Let T be the right endpoint of the interval of existence. What happens at T to prevent the solution from extending further?

c.) Solve the IVP analytically to solve for the exact value of T. [You can use *Wolfram|Alpha* or something else to help you find this number.] How close was your approximation?

d.) The Runge-Kutta method continues to give you y-values for $t > T$. Do these values have any significance? If so, what? If not, why?

Let's take a look at the error generated in the improved Euler method. If such a method is to be used in practice, then we need to know how small we have to choose h to get an approximation that is "close enough" to the actual solution. On the other hand, we don't want to do more work than we absolutely need to. Thus we should choose the optimal value for h that gives us the accuracy we want, but doesn't make us do any more work than we need to.

Recall that the improved Euler formula for y_{k+1} is

$$y_{k+1} = y_k + \left(\frac{h}{2}\right)(f_k + f(t_{k+1}, y_k + h \cdot f_k),$$

where $f_k = f(t_k, y_k)$.

Suppose that the exact solution φ is at least three-times differentiable. We want to show that the local truncation error for the improved Euler method is proportional to h^3. By assuming that φ is C^3, we are able to write a Taylor expansion for φ around t_k up to third order. We get

$$\varphi(t_{k+1}) = \varphi(t_k + h) = \varphi(t_k) + \varphi'(t_k) \cdot h + \frac{\varphi''(t_k)}{2!} \cdot h^2 + \frac{\varphi'''(t_k^*)}{3!} \cdot h^3, \tag{1}$$

where $t_k < t_k^* < t_{k+1}$. Now assume that $y_k = \varphi(t_k)$.

3. [10] Show that the error generated by the $(k+1)$-st step is given by

$$\begin{aligned} e_{k+1} &= \varphi(t_{k+1}) - y_{k+1} \\ &= \frac{\varphi''(t_k)h - [f(t_{k+1}, y_k + h \cdot f_k) - f_k]}{2!} \cdot h + \frac{\varphi'''(t_k^*)}{3!} \cdot h^3 \end{aligned}$$

4. [10] Using the facts that $\varphi'' = f_t(t, \varphi(t)) + f_y(t, \varphi(t))\varphi'(t)$ and the Taylor approximation with a remainder for a function $F(t, y)$ of two variables is

$$F(a+h, b+k) = F(a,b) + F_t(a,b) \cdot h + F_y(a,b) \cdot k + \frac{1}{2!}\left[h^2 F_{tt} + 2hkF_{ty} + k^2 F_{yy}\right]\Big|_{(\xi,\eta)},$$

where $\xi \in (a, a+h)$ and $\eta \in (b, b+k)$, show that the first term on the right-hand side of the equation for e_{k+1} is proportional to h^3 plus higher order terms. This is the desired result.

5. [10] Show that if $f(t, y)$ is linear in both t and y, then $e_{k+1} = \dfrac{h^3 \varphi'''(t_k^*)}{6}$ for some $t_k^* \in (t_k, t_{k+1})$. [Hint: What are f_{tt}, f_{ty}, and f_{yy}?]

6. [10] Consider the DE $y' - 2y = 3t$ with initial condition $y(0) = 3$. Calculate the maximum error e_{10} with $h = 0.05$.