Math 555: Differential Equations
Good Problems 8

Due: Thursday, 24 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 5 problems.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:________________________________

1. Use the definition to calculate the Laplace transform $\mathcal{L}\{f(t)\}$ where

$$f(t) = u_c(t), \quad c > 0.$$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt$$

$$= \int_0^c e^{-st} \cdot 0\,dt + \int_c^\infty e^{-st} \cdot 1\,dt$$

$$= \int_c^\infty e^{-st}\,dt$$

$$= \lim_{A\to\infty} \int_c^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} \right]_c^A$$

$$= \lim_{A\to\infty} -\frac{1}{s} e^{-sA} + \frac{1}{s} e^{-cs}$$

$$= \frac{e^{-cs}}{s} \quad \text{when } s > 0 \quad \square$$

2. Use the <u>definition</u> to calculate the Laplace transform $\mathcal{L}\{f(t)\}$ where

$$f(t) = u_{\frac{\pi}{2}}(t)\cos(t).$$

$$\mathcal{L}\{u_{\frac{\pi}{2}}(t)\cos(t)\} = \int_0^\infty e^{-st} u_{\frac{\pi}{2}}(t) \cos t \, dt$$

$$= \cancel{\int_0^{\frac{\pi}{2}} e^{-st} \cdot 0 \cdot \cos t \, dt}^{\,0} + \int_{\frac{\pi}{2}}^\infty e^{-st} \cdot 1 \cdot \cos t \, dt$$

$$= \underbrace{\int_{\frac{\pi}{2}}^\infty e^{-st} \cos t \, dt}$$

$$= \int_{\frac{\pi}{2}}^\infty e^{-st} \frac{d}{dt}[\sin t] \, dt$$

$$= e^{-st} \sin t \Big|_{\frac{\pi}{2}}^\infty + s \int_{\frac{\pi}{2}}^\infty e^{-st} \frac{d}{dt}[-\cos t] \, dt$$

$$= e^{-st} \sin t \Big|_{\frac{\pi}{2}}^\infty - s e^{-st} \cos t \Big|_{\frac{\pi}{2}}^\infty - s^2 \underbrace{\int_{\frac{\pi}{2}}^\infty e^{-st} \cos t \, dt}$$

$$\Rightarrow \quad (1+s^2) \int_{\frac{\pi}{2}}^\infty e^{-st} \cos t \, dt = \lim_{A\to\infty} \left[e^{-sA} \sin A - e^{-\frac{\pi}{2}s} \underbrace{\cancel{\sin\left(\tfrac{\pi}{2}\right)}}_{1} - s e^{-As} \cos A + \underbrace{s e^{-\frac{\pi}{2}s} \cancel{\cos\left(\tfrac{\pi}{2}\right)}}_{=0} \right]$$

$$= -e^{-\pi/2 s}, \quad s > 0$$

limits $= 0$ if $s > 0$
[use squeeze thm]

$$\Rightarrow \quad \boxed{\mathcal{L}\{u_{\frac{\pi}{2}}(t)\cos(t)\} = \int_{\frac{\pi}{2}}^\infty e^{-st} \cos t \, dt = \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}, \quad s > 0}$$

Name: ______________________

3. Use Laplace transforms to solve the initial value problem

$$\begin{cases} y'' + 4y = \sin t - u_{2\pi}(t)\sin(t-2\pi); \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

Apply $\mathcal{L}$:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\}$$

$$s^2\mathcal{L}\{y\} - s\underbrace{\cancel{y(0)}}_{0} - \underbrace{\cancel{y'(0)}}_{0} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} - e^{-2\pi s}\mathcal{L}\{\sin t\}$$

by initial cond.

$$(s^2+4)Y(s) = \frac{1-e^{-2\pi s}}{s^2+1} \Rightarrow Y(s) = \frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}$$

Now for the fun part ☺ Put $H(s) = \frac{1}{(s^2+4)(s^2+1)}$ so that

$Y(s) = H(s) - e^{-2\pi s}H(s)$. Then $y(t) = h(t) - u_{2\pi}(t)h(t-2\pi)$,

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

$$H(s) = \frac{1}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} = A\frac{s}{s^2+4} + \frac{B}{2}\cdot\frac{2}{s^2+4} + C\frac{s}{s^2+1} + D\frac{1}{s^2+1}$$

So $h(t) = A\cos(2t) + \frac{B}{2}\sin(2t) + C\cos t + D\sin t$

Now we just need to find the coefficients!

$$1 = (As+B)(s^2+1) + (Cs+D)(s^2+4) = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

$$\Rightarrow \begin{cases} A+C=0 \\ A+4C=0 \end{cases} \quad \boxed{A=C=0} \qquad \text{and} \qquad \begin{cases} B+D=0 \\ B+4D=1 \end{cases}$$

$3D = 1$

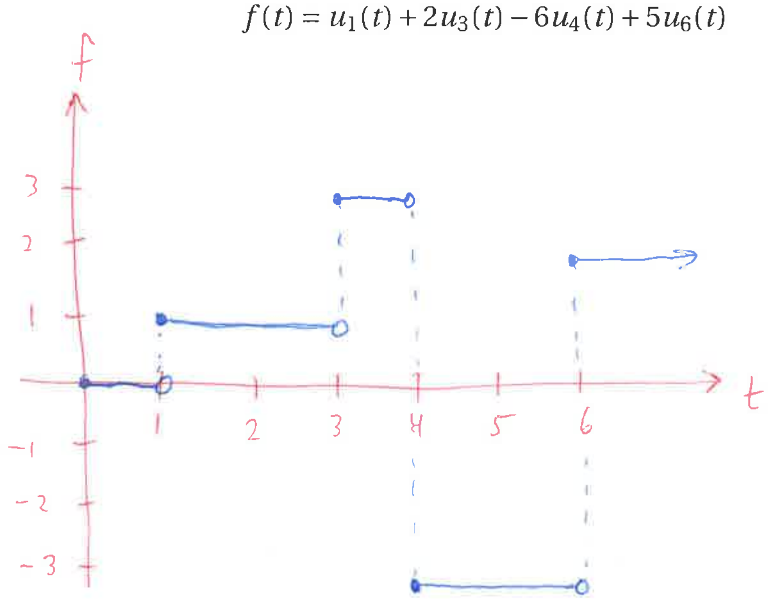
$D = \frac{1}{3}$, $B = -\frac{1}{3}$

Plug in to obtain the final sol'n:

$$\boxed{\begin{array}{l} y(t) = h(t) - u_{2\pi}(t)\,h(t-2\pi) \text{ where} \\ h(t) = -\frac{1}{6}\sin(2t) + \frac{1}{3}\sin t \end{array}}$$

4. Sketch the graph of the function and find its Laplace transform.

$$f(t) = u_1(t) + 2u_3(t) - 6u_4(t) + 5u_6(t)$$



$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\} + 2\mathcal{L}\{u_3(t)\} - 6\,\mathcal{L}\{u_4(t)\} + 5\,\mathcal{L}\{u_6(t)\}$$

$$= \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s} + 5\frac{e^{-6s}}{s}$$

$$= \frac{e^{-s} + 2e^{-3s} - 6e^{-4s} + 5e^{-6s}}{s}$$

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5. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}.$$

$$F(s) = e^{-s}\frac{s-2}{s^2-4s+3} = e^{-s}\frac{s-2}{(s-3)(s-1)} = e^{-s}\left[\frac{A}{s-3}+\frac{B}{s-1}\right]$$

Recall that $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$

PFD:

$$s-2 = A(s-1) + B(s-3)$$

$s=1$: $\quad -1 = -2B \Rightarrow B = \frac{1}{2}$

$s=3$: $\quad 1 = 2A \Rightarrow A = \frac{1}{2}$

So $\quad F(s) = \frac{1}{2}e^{-s}\frac{1}{s-3} + \frac{1}{2}e^{-s}\frac{1}{s-1}$

$$\Rightarrow f(t) = \frac{1}{2}u_1(t)e^{3(t-1)} + \frac{1}{2}u_1(t)e^{t-1}$$

$$\boxed{f(t) = \frac{e^{-3}}{2}u_1(t)e^{3t} + \frac{e^{-1}}{2}u_1(t)e^{t}}$$