Math 555: Differential Equations
Good Problems 8.5

Due: Friday, 25 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all problems.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Find the general solution of the differential equation

$$(2y + x^2 y)\, y' = x^2.$$

Classification: First-order separable!

$$x^2+2 \overline{\smash{\big)}\, x^2} \quad 1 - \frac{2}{x^2+2}$$
$$\frac{x^2+2}{-2}$$

$$y\,dy = \frac{x^2}{2+x^2}\,dx$$

$$\frac{1}{2}y^2 = \int 1 - \frac{2}{x^2+2}\,dx = \int 1 - \frac{1}{(\frac{1}{2}x^2+1)}\,dx = \int 1 - \frac{1}{(\frac{1}{\sqrt{2}}x)^2+1}\,dx$$

$$= x - \sqrt{2}\int \frac{1}{u^2+1}\,du = x - \sqrt{2}\tan^{-1}(u) + C$$

$$= x - \sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}x\right) + C$$

$$\left(\begin{array}{l} u = \frac{1}{\sqrt{2}}x \\ du = \frac{1}{\sqrt{2}}dx \\ \sqrt{2}\,du = dx \end{array} \right)$$

so $$\boxed{y^2 = 2x - 2\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}x\right) + C}$$

2. Find the general solution of the differential equation

$$y' + \frac{1}{t}y = t^2 \sin t.$$

Classification: First-order Linear!

u	dv
$+s^3$	$\sin s$
$-3s^2$	$-\cos s$
$+6s$	$-\sin s$
-6	$\cos s$
$+0$	$\sin s$

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t$$

$$y(t) = \frac{1}{t}\int_{t_0}^{t} s \cdot s^2 \sin(s)\,ds + \frac{c}{t}$$

$$= \frac{1}{t}\int_{t_0}^{t} s^3 \sin(s)\,ds + \frac{c}{t}$$

$$= \frac{1}{t}\left[-t^3\cos t + 3t^2\sin t + 6t\cos t - 6\cos t\right] + \frac{c}{t}$$

$$\boxed{y = -t^2\cos t + 3t\sin t + 6\cos t - \frac{6}{t}\cos t + \frac{c}{t}}$$

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3. Consider the differential equation

$$x^2y'' + xy' - 4y = 0, \quad x > 0.$$

(a.) Show that $y_1(x) = x^2$ is a solution of the equation.

(b.) Use the method of reduction of order to find a second solution y_2.

(c.) Find the Wronskian of y_1 and y_2.

(a.) $\left.\begin{array}{l} y_1 = x^2 \\ y_1' = 2x \\ y_1'' = 2 \end{array}\right\}$ $x^2y_1'' + xy_1' - 4y_1 = 2x^2 + 2x^2 - 4x^2 = 0$ $\square$

(b.) Put: $y = v(x)\cdot x^2$

Calculate: $y' = 2vx + v'x^2$

$y'' = 2v + 2v'x + 2v'x + v''x^2 = 2v + 4v'x + v''x^2$

Plug in: $x^2(2v + 4v'x + v''x^2) + x(2vx + v'x^2) - 4vx^2 = 0$

$x^4v'' + v'(4x^3 + x^3) + v(\underbrace{\cancel{2x^2 + 2x^2 - 4x^2}}_{0}) = 0$

$\Rightarrow x^4v'' + 5x^3v' = 0$

or $v'' = \frac{-5}{x}v' \Rightarrow \frac{dv'}{v'} = \frac{-5}{x}dx$

$\Rightarrow \ln v' = -5\ln x + C_2 \Rightarrow v' = C_2x^{-5} \Rightarrow v(x) = C_2x^{-4} + C_1$

so $y = C_1x^2 + C_2x^{-2}$ and $\boxed{y_2(x) = x^{-2}}$

(c) $W(y_1,y_2) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1} = \frac{-4}{x} \neq 0$ whenever $x \neq 0$.

4. Solve the initial value problem

$$\begin{cases} y' + 2ty = 4t^3; \\ y(0) = 1. \end{cases}$$

Classification: First-order Linear!

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}$$

$$y(t) = e^{-t^2} \int_0^t e^{s^2} 4s^3\,ds + 1\, e^{-t^2}$$

$$= e^{-t^2}\left[2t^2 e^{t^2} - 2e^{t^2} + 2\right] + e^{-t^2}$$

$$\boxed{y = 2t^2 - 2 + 3e^{-t^2}}$$

$$\int_0^t 4s^3 e^{s^2}\,ds \quad \left\{\begin{matrix} u = s^2 \\ du = 2s\,ds \end{matrix}\right\}$$

$$\to \int_0^{t^2} 2ue^u\,du$$

$$= 2ue^u - 2e^u \Big|_0^{t^2}$$

$$= 2t^2e^{t^2} - 2e^{t^2} + 2$$

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5. Solve the initial value problem

$$\begin{cases} (2x-y)-(x-2y)y' = 0; \\ y(1) = 3. \end{cases}$$

Classification: First-order not linear; not separable; exact??

$$M = 2x - y \qquad \partial_y M = -1$$
$$N = -x + 2y \qquad \partial_x N = -1$$
Exact!

$$\int M\,dx = \int 2x - y\,dx = x^2 - xy + h(y)$$
$$\int N\,dy = \int -x + 2y\,dy = y^2 - xy + g(x)$$
$$\psi(x,y) = x^2 - xy + y^2 = C$$

$$C = 1^2 - 3\cdot 1 + 3^2 = 1 - 3 + 9 = 7$$

so the solution is

$$\boxed{x^2 - xy + y^2 = 7}$$

Now, solve for y: $y^2 - xy + x^2 - 7 = 0$

$$y = \frac{x \pm \sqrt{x^2 - 4(x^2-7)}}{2}$$

at $x = 1$: $\dfrac{1 \pm \sqrt{1 - 4(-6)}}{2} = \dfrac{1 \pm 5}{2}$: 3 ✓, -2

so $$y = \frac{x + \sqrt{x^2 - 4(x^2-7)}}{2}$$

or $$\boxed{y = \frac{x}{2} + \frac{1}{2}\sqrt{28 - 3x^2}}$$ *

6. Find the general solution of the DE

$$x^2 y'' - 3xy' + 4y = x^2 \ln(x).$$

[Hint: First solve the associated homogeneous equation.]

Homog.: $x^2 y'' - 3xy' + 4y = 0$ Euler!

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2, 2$$

$$y(x) = C_1 x^2 + C_2 x^2 \ln x$$

so $y_1 = x^2$, $y_2 = x^2 \ln x$

$$g(x) = \frac{x^2 \ln x}{x^2} = \ln x$$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3$$

Variation of Parameters:

$$y = -y_1(x) \int_{x_0}^{x} \frac{y_2(s)\, g(s)}{W(s)}\, ds + y_2(x) \int_{x_0}^{x} \frac{y_1(s)\, g(s)}{W(s)}\, ds$$

$$= -x^2 \int_{x_0}^{x} \frac{s^2 (\ln s)^2}{s^3}\, ds + x^2 \ln x \int_{x_0}^{x} \frac{s^2 \ln s}{s^3}\, ds$$

$$= -x^2 \int_{x_0}^{\ln x} u^2\, du + x^2 \ln x \int_{x_0}^{\ln x} u\, du$$

$$= -x^2 \left(\tfrac{1}{3} (\ln x)^3 + C_1 \right) + x^2 \ln x \left(\tfrac{1}{2} (\ln x)^2 + C_2 \right)$$

$$y(x) = C_1 x^2 + C_2 x^2 \ln x - \frac{x^2}{3} (\ln x)^3 + \frac{x^2}{2} (\ln x)^3$$

so $\boxed{y(x) = C_1 x^2 + C_2 x^2 \ln x + \frac{x^2}{6} (\ln x)^3}$

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7. Use the method of undetermined coefficients to solve the initial value problem

$$\begin{cases} y'' - 2y' + y = te^t + 4; \\ y(0) = 1, \\ y'(0) = 1. \end{cases}$$

Classification: second-order linear constant coeff. nonhomog.

$$\left.\begin{aligned} &y'' - 2y' + y = 0 \\ &r^2 - 2r + 1 = 0 \\ &(r-1)^2 = 0 \\ &r = 1, 1 \end{aligned}\right\} \quad \text{homog. sol'n: } \; y = C_1e^t + C_2te^t$$

Undetermined Coeff.: homog!

$$Y(t) = t^2(At+B)e^t + C = At^3e^t + Bt^2e^t + C$$

$$Y'(t) = 3At^2e^t + At^3e^t + 2Bte^t + Bt^2e^t + 0$$

$$= At^3e^t + (3A+B)t^2e^t + 2Bte^t$$

$$Y''(t) = 3At^2e^t + At^3e^t + 2(3A+B)te^t + (3A+B)t^2e^t + 2Be^t + 2Bte^t$$

$$= At^3e^t + (6A+B)t^2e^t + (6A+4B)te^t + 2Be^t$$

$$Y'' - 2Y' + Y = e^t\left[\cancel{At^3} + \cancel{(6A+B)t^2} + (6A+4B)t + 2B - \cancel{2At^3} - \cancel{(6A+2B)t^2} - 4Bt + \cancel{At^3} + \cancel{Bt^2}\right] + C$$

so $C = 4$, $B = 0$, $A = \frac{1}{6}$ and

$$\boxed{y(t) = \frac{1}{6}t^3e^t + 4 + C_1e^t + C_2te^t}$$

8. Find the general solution of the DE

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}.$$

Need more info!
Must know y_1 and y_2 to use Variation of Parameters...

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9. (*a*.) Find the Taylor series for $f(x) = \sin(x - \pi/4)$ centered at $x_0 = \pi/4$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad x_0 = \frac{\pi}{4}$$

$f(x) = \sin(x - \pi/4)$ $\quad f(\pi/4) = \sin(0) = 0$

$f'(x) = \cos(x - \pi/4)$ $\quad f'(\pi/4) = \cos 0 = 1$

$f''(x) = -\sin(x - \pi/4)$ $\quad f''(\pi/4) = -\sin 0 = 0$

$f'''(x) = -\cos(x - \pi/4)$ $\quad f'''(\pi/4) = -\cos 0 = -1$

$f^{(4)}(x) = \sin(x - \pi/4)$ $\quad f^{(4)}(\pi/4) = \sin 0 = 0$

$\vdots \qquad \vdots$

$f^{(n)}(\pi/4) =$ 2 cases

even: $f^{(2n)}(\pi/4) = 0$

odd: $f^{(2n+1)}(\pi/4) = (-1)^n$

So,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x - \pi/4)^{2n+1}}$$

(*b*.) Find the radius of convergence.

$$\sum_{n=o}^{\infty} \frac{(-1)^n n^3 (x-3)^n}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(-1)^{n+1}}(n+1)^3}{2^{n+1}} \cdot \frac{\cancel{2^n}}{\cancel{(-1)^n}\, n^3}\right| = \left|\frac{n^3 + 3n^2 + 3n + 1}{2n^3}\right| \xrightarrow{n\to\infty} \frac{1}{2}$$

So $\boxed{R = \left(\frac{1}{2}\right)^{-1} = 2}$

10. Suppose you wish to find a power series solution to the DE

$$(2+x^2)y'' - xy' + 4y = 0.$$

Find the recurrence relation. Do <u>NOT</u> solve the DE.

$x_0 = 0$ is ordinary, so center there.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(2+x^2)y'' - xy' + 4y = (2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+1)(n+2)a_{n+2} x^n$$

$$4a_2 + 12a_3 x + \sum_{n=2}^{\infty} 2(n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n + 4a_0 + 4a_1 x + \sum_{n=2}^{\infty} 4a_n x^n = 0$$

$$= (4a_0 + 4a_2) + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty}\left[2(n+1)(n+2)a_{n+2} + n(n-1)a_n - n a_n + 4a_n\right]x^n = 0$$

$\boxed{a_2 = -a_0}$ $\boxed{a_3 = -\frac{1}{4}a_1}$ $\boxed{a_{n+2} = \frac{-(n^2-2n+4)a_n}{2(n+1)(n+2)}, \ n \geq 2}$