Math 555: Differential Equations
Good Problems 7

Due: Friday, 18 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 6 problems.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:________________________________

1. Let $\mathcal{L}\{f\}$ denote the Laplace transform of f. Use the definition of the Laplace transform to show that

1. $\mathcal{L}\{\alpha f\} = \alpha\mathcal{L}\{f\}$ for all $\alpha \in \mathbb{R}$, and
2. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$.

This tells us that $\mathcal{L}$ is a *linear operator.*

1.) $\mathcal{L}\{\alpha f\} = \int_0^\infty e^{-st}(\alpha f(t))\,dt = \alpha \int_0^\infty e^{-st} f(t)\,dt = \alpha\,\mathcal{L}\{f\}.$

2.) $\mathcal{L}\{f+g\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt = \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$

$= \mathcal{L}\{f\} + \mathcal{L}\{g\}.$ $\square$

2. Use the *definition* of the Laplace transform to calculate $F(s) = \mathcal{L}\{\cos(at)\}$, $a \in \mathbb{R}$. Be sure to state the domain of F.

Two ways:

1. Integrate by parts twice as we did for $\sin(at)$ in class.

2. write $\cos(at) = \frac{1}{2}\left(e^{iat} + e^{-iat}\right)$

Using 2.)

$$\mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt = \int_0^\infty e^{-st}\,\frac{1}{2}\left(e^{iat} + e^{-iat}\right)dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-ia)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+ia)t}\,dt$$

$$= \frac{1}{2}\left[\frac{-1}{s-ia}e^{-(s-ia)t}\Big|_0^\infty + \frac{-1}{s+ia}e^{-(s+ia)t}\Big|_0^\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] - \frac{1}{2}\lim_{A\to\infty}\cancel{\left(\frac{1}{s-ia}e^{-(s-ia)A}\right)}^{\,0,\ s>0} + \frac{1}{2}\lim_{A\to\infty}\cancel{\frac{1}{s+ia}e^{-(s+ia)A}}^{\,0,\ s>0}$$

$$= \frac{1}{2}\left(\frac{s+ia+s-ia}{s^2+a^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2+a^2}\right) = \boxed{\frac{s}{s^2+a^2},\quad s>0}$$

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3. Prove the theorem:

Theorem. *Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Pf.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st} \frac{d}{dt}[f(t)]\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{A\to\infty} e^{-sA} f(A) - f(0) + s\mathcal{L}\{f(t)\}.$$

for $t \ge M$ $f(A) \le |f(A)| \le Ke^{aA}$

so $\lim_{A\to\infty} e^{-sA} f(A) \le \lim_{A\to\infty} K e^{-(s-a)A} = 0$ when $s > a$.

Thus $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ for $s > a$. ∎

4. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for each of the functions.

(*a*.) $F(s) = \dfrac{2s+2}{s^2+2s+5} = \dfrac{2s+2}{(s^2+2s+1)+4} = \dfrac{2s+2}{(s+1)^2+2^2} = 2\,\dfrac{(s+1)}{(s+1)^2+2^2}$

$\boxed{\mathcal{L}^{-1}\{F\} = 2e^{-t}\cos 2t}$ by the table.

(*b*.) $F(s) = \dfrac{3s}{s^2-s-6} = \dfrac{A}{s-3} + \dfrac{B}{s+2} = \dfrac{9}{5}\dfrac{1}{s-3} + \dfrac{6}{5}\dfrac{1}{s+2}$

$s^2-s-6 = (s-3)(s+2)$

$3s = A(s+2) + B(s-3)$

$s=-2$: $\quad -6 = -5B \quad \Rightarrow \quad B = \frac{6}{5}$

$s=3$: $\quad 9 = 5A \quad \Rightarrow \quad A = 9/5$

$\boxed{\mathcal{L}\{F\} = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}}$ by the table.

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5. Use the Laplace transform to solve the initial value problem

$$\begin{cases} y^{(4)} - 4y = 0; \\ y(0) = 1, \\ y'(0) = 0, \\ y''(0) = 1, \\ y'''(0) = 0. \end{cases}$$

$$\mathcal{L}\{y^{(4)} - 4y\} = s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 4\mathcal{L}\{y\}$$

$$= s^4 Y - 4Y - s^3 - s = 0$$

$$\Rightarrow \quad Y(s) = \frac{s^3+s}{s^4-4} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2-2}$$

$$\Rightarrow \quad s^3 + s = (As+B)(s^2-2) + (Cs+D)(s^2+2)$$

$$= As^3 + Bs^2 - 2As - 2B + Cs^3 + Ds^2 + 2Cs + 2D$$

$$= (A+C)s^3 + (B+D)s^2 + (2C-2A)s + (2D-2B)$$

$$A + C = 1 \qquad B + D = 0 \quad \Rightarrow \quad B = D = 0.$$

$$-2A + 2C = 1 \quad , \quad -2B - 2D = 0 \quad \Rightarrow \quad B = D$$

$$4C = 3$$

$$\Rightarrow C = \frac{3}{4}$$

$$\Rightarrow A = \frac{1}{4}$$

So $Y(s) = \frac{1}{4}\frac{s}{s^2+2} + \frac{3}{4}\frac{s}{s^2-2}$

and by the table,

$$\boxed{y(t) = \frac{1}{4}\cos(\sqrt{2}t) + \frac{3}{4}\cos(\sqrt{2}t)}$$

6. Prove the theorem:

Theorem. *If* $F(s) = \mathcal{L}\{f(t)\}$ *exsits for* $s > a \geq 0$, *and if* $c \in \mathbb{R}$ *is a constant, then*

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \qquad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, *then* $e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$.

pf.

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\,dt = \int_0^\infty e^{-(s-c)t} f(t)\,dt$$

let $u = s - c$:

$$= \int_0^\infty e^{-ut} f(t)\,dt = F(u) = F(s-c). \quad \square$$