Math 555: Differential Equations
Good Problems 6

Due: Friday, 11 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:________________________________

1. Determine the radii and intervals of convergence of the given power series.

(a.) $\sum_{n=0}^{\infty} \frac{n}{2^n}(x-3)^n$

Ratio Test: $\left| \frac{2^n}{n} \cdot \frac{n+1}{2^{n+1}} \right| = \frac{1}{2} \left| \frac{n+1}{n} \right| \xrightarrow{n\to\infty} \frac{1}{2}$

Thus the radius of convergence is $\boxed{R=2}$

$3-2=1$

$3+2=5$

Thus, the interval of convergence is $(1,5)$

At $x=1$:

$\sum_{n=0}^{\infty} \frac{n}{2^n}(1-3)^n = \sum_{n=0}^{\infty} \frac{n}{2^n}(-2)^n = \sum_{n=0}^{\infty} (-1)^n n$ diverges by the alt. series test ($n \not\to 0$ as $n\to\infty$.)

At $x=5$

$\sum_{n=0}^{\infty} \frac{n}{2^n}(5-3)^n = \sum_{n=0}^{\infty} n$ diverges by many tests including integral.

! → (b.) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

Ratio: $\left| \frac{\cancel{(-1)^{n+1}} (n+1)^2}{3^{n+1}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n} n^2} \right| = \frac{1}{3} \left| \frac{(n+1)^2}{n^2} \right| = \frac{1}{3} \left(\frac{n+1}{n} \right)^2 \to \frac{1}{3}$ as $n\to\infty$.

Thus, the radius of convergence is $\boxed{R=3}$

Endpoints: $x=-2-3=-5$:

$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-5+2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^n} n^2 \cancel{(-3)^n}}{\cancel{3^n}} = \sum_{n=1}^{\infty} n^2$ diverges.

$x=-2+3=1$:

$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (1+2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n n^2 \cancel{3^n}}{\cancel{3^n}} = \sum_{n=1}^{\infty} (-1)^n n^2$ diverges.

So the interval of convergence is $(-5,1)$

2. Find the Taylor series expansions about the given point x_0 for the given function. Also determine the radius of convergence.

(*a*.) $\ln(x)$, $x_0 = 1$.

This is nonsense "^ sorry!

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \frac{2\cdot 1}{x^3} \qquad f'''(1) = 2\cdot 1$

$f^{(4)}(x) = \frac{-3\cdot 2\cdot 1}{x^4} \qquad f^{(4)}(1) = -3\cdot 2\cdot 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad f^{(n)}(1) = (-1)^{n+1}(n-1)!$

So $\cancel{\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(n-1)!\,(x-1)^n}$

sorry "

$\cancel{\left|\frac{(-1)^{n+2}\, n!}{(-1)^{n+1}(n-1)!}\right| = |n| \to \infty \text{ as } n \to \infty}$

~~so R = 0~~ "

Taylor's Formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

So,

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

or

$$\boxed{\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$$

Ratio Test:

$$\left|\frac{(-1)^{n+2}}{n+1} \cdot \frac{n}{(-1)^{n+1}}\right| = \left|\frac{n}{n+1}\right| \to 1 \text{ as } n \to \infty$$

So $\boxed{R = 1}$

(*b*.) $\dfrac{1}{1-x}$, $x_0 = 2$.

$$\frac{1}{1-x} = \frac{1}{1-(x-2)+2} = \frac{1}{3-(x-2)} = \frac{1/3}{1-\left(\frac{x-2}{3}\right)}$$

So
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{x-2}{3}\right)^n = \boxed{\sum_{n=0}^{\infty} 3^{-n-1}(x-2)^n}$$

$\boxed{R = 3}$ by geometric series.

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3. Determine the general term a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + 2a_n\right] x^n = 0$$

So $(n+1)a_{n+1} + 2a_n = 0$

$$\Rightarrow \boxed{a_{n+1} = \frac{-2a_n}{n+1}}$$

recurrence relation

a_0 = unknown

$a_1 = -2a_0$

$a_2 = \frac{-2a_1}{2} = \frac{-2\cdot(-2a_0)}{2} = \frac{(-1)^2 2^2 a_0}{2\cdot 1}$

$a_3 = \frac{-2a_2}{3} = \frac{(-1)^3 2^3 a_0}{3\cdot 2\cdot 1}$

$\vdots$

$a_n = \frac{(-1)^n 2^n a_0}{n!}$

So,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n a_0}{n!} x^n$$

$$= \boxed{a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n}$$

$$\ldots = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}(-2x)^n$$

$$= \boxed{a_0 e^{-2x}}$$

4. Seek to find a power series solution to the given DE, centered about the given point.

(*a*.) Find the recurrence relation.

(*b*.) Find the first four terms in each of the two solutions y_1 and y_2, unless the series terminates sooner.

(*c*.) If possible, find the general term.

$$y'' - xy' - y = 0, \quad x_0 = 0.$$

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}\right\} \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n + 2a_2 - a_0 = 0$$

a) Thus, $\boxed{a_2 = \frac{a_0}{2}}$ and $\boxed{a_{n+2} = \frac{\cancel{(n+1)}\, a_n}{(n+2)\cancel{(n+1)}}, \ n \geq 1.}$

$a_0 = ?$

$a_2 = \frac{a_0}{2}$

$a_4 = \frac{a_2}{4} = \frac{a_0}{4 \cdot 2}$

$a_6 = \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2}$

$\vdots$

$a_{2n} = \frac{a_0}{2(n!)}$

$a_1 = ?$

$a_3 = \frac{a_1}{3}$

$a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3}$

$a_7 = \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3}$

$\vdots$

$a_{2n+1} = \frac{2 a_1 n!}{(2n+1)!}$

b) $$\boxed{\begin{aligned} y_1 &= a_0 + \frac{a_0}{2} x^2 + \frac{a_0}{4 \cdot 2} x^4 + \frac{a_0}{6 \cdot 4 \cdot 2} x^6 + \cdots \\ y_2 &= a_1 x + \frac{a_1}{3} x^3 + \frac{a_1}{5 \cdot 3} x^5 + \frac{a_1}{7 \cdot 5 \cdot 3} x^7 + \cdots \end{aligned}}$$

c) $$\boxed{y = a_0 \sum_{n=0}^{\infty} \frac{1}{2(n!)} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2(n!)}{(2n+1)!} x^{2n+1}}$$

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5. Seek to find a power series solution to the given DE, centered about the given point.

(*a.*) Find the recurrence relation.

(*b.*) Find the first four terms in each of the two solutions y_1 and y_2, unless the series terminates sooner.

(*c.*) If possible, find the general term.

$$(3-x^2)y'' - 3xy' - y = 0, \quad x_0 = 0.$$

$y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$(3-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

$n^2 - n + 3n + 1 = n^2 + 2n + 1 = (n+1)^2$

$\Longrightarrow \sum_{n=2}^{\infty}\left[3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n - a_n\right]x^n + 6a_2 + 18a_3x - 3a_1x - a_0 - a_1x = 0$

a) So, $\boxed{6a_2 - a_0 = 0}$, $\boxed{18a_3 - 4a_1 = 0}$, and $\boxed{a_{n+2} = \frac{(n+1)^2 a_n}{3(n+2)(n+1)} = \frac{(n+1)a_n}{3(n+2)}}$

$a_0 = ?$

$a_2 = \frac{a_0}{6}$

$a_4 = \frac{3a_2}{3\cdot 4} = \frac{3\cdot a_0}{3^2\cdot 4\cdot 2\cdot 1} = \frac{3\cdot a_0}{3^2\cdot 4\cdot 2}$

$a_6 = \frac{5a_4}{3\cdot 6} = \frac{5\cdot 3\cdot 1\, a_0}{3^3\cdot 6\cdot 4\cdot 2}$

$a_1 = ?$

$a_3 = \frac{2a_1}{3\cdot 3\cdot 1}$

$a_5 = \frac{4a_3}{3\cdot 5} = \frac{4\cdot 2\, a_1}{3^2 5\cdot 3\cdot 1}$

$a_7 = \frac{6\cdot a_5}{3\cdot 7} = \frac{6\cdot 4\cdot 2\, a_1}{3^3\cdot 7\cdot 5\cdot 3\cdot 1}$

$\vdots$

$a_{2n} = \frac{(n-1)!\, a_0}{3^n\cdot 2^2\cdot (n!)^2}$

$a_{2n+1} = \frac{2^2 (n!)^2 a_1}{3^n\cdot (2n+1)!}$

so the sol'n is

$$\boxed{y = a_0 \sum_{n=0}^{\infty} \frac{(n-1)!}{3^n 2^2 (n!)^2} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^2 (n!)^2}{3^n\cdot(2n+1)!} x^{2n+1}}$$

6. Find a series solution of the initial value problem.

$$\begin{cases} y'' + x^2 y = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

$$\left.\begin{array}{l} y = \sum_{n=0}^{\infty} a_n x^n \\ y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{array}\right\} \Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-2}\right] x^n + 2a_2 + 6a_3 x = 0$$

so, $\boxed{2a_2 = 0}$, $\boxed{6a_3 = 0}$, and $\boxed{a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}}$

a_0	a_1	$a_2 = 0$	$a_3 = 0$
$a_4 = \frac{-a_0}{4 \cdot 3}$	$a_5 = \frac{-a_1}{5 \cdot 4}$	$a_6 = 0$	$a_7 = 0$
$a_8 = \frac{-a_4}{8 \cdot 7} = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3}$	$a_9 = \frac{-a_5}{9 \cdot 8} = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4}$	$a_{10} = 0$	$a_{11} = 0$
$a_{12} = \frac{-a_8}{12 \cdot 11} = \frac{-a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$	$a_{13} = \frac{-a_9}{13 \cdot 12} = \frac{-a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$	$a_{14} = 0$	$a_{15} = 0$
		$\vdots$	$\vdots$

$$y_1 = a_0 - \frac{a_0}{4 \cdot 3} x^4 + \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3} x^8 - \frac{a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{12} + \cdots$$

$$y_2 = a_1 x - \frac{a_1}{5 \cdot 4} x^5 + \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4} x^9 - \frac{a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{13} + \cdots$$

Now use the initial conditions: $a_0 = 1$ and $a_1 = 0$.

The soln is:

$$\boxed{y = 1 - \frac{1}{12} x^4 + \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} x^8 - \frac{1}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{12} + \cdots}$$

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7. Determine a lower bound for the radius of convergence of series solutions about the given point x_0 for each of the differential equations, *without solving the DE.*

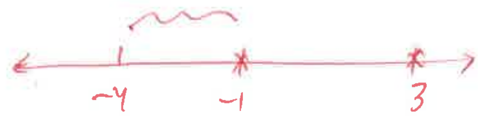
(*a.*) $(x^2-2x-3)y''+xy'+4y=0, \quad x_0=-4;$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3,-1$ are the singular pts.

$\boxed{R=3}$



(*b.*) $(x^2-2x-3)y''+xy'+4y=0, \quad x_0=4;$

$x=3,-1$ are the singular pts again

$\boxed{R=1}$

(*c.*) $(1+x^3)y''+4xy'+y=0, \quad x_0=2.$

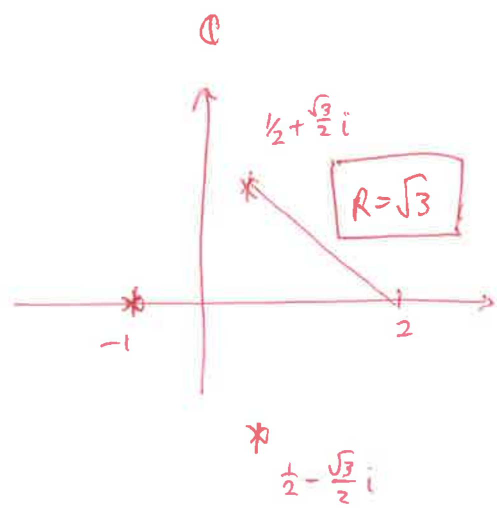
$x^3+1=(x+1)(x^2-x+1)=0$

$x=-1 \qquad (x-1/2)^2+3/4=0$

$x=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

So $-1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$ are

the singular points in $\mathbb{C}$.



8. Find the general solution of the given DE that is valid in any interval that is valid in any interval not including the singular point.

(*a.*) $x^2y'' + 4xy' + 2y = 0$;

Euler!

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -1, -2$

so $\boxed{y = C_1 x^{-1} + C_2 x^{-2}, \quad x \neq 0.}$

(*b.*) $(x-2)^2y'' + 5(x-2)y' + 8y = 0$.

$u = x-2$: $\quad u^2 y'' + 5uy' + 8y = 0$

$r^2 + 4r + 8 = 0$

$(r+2)^2 = -4$

$r = -2 \pm 2i$

so $y(u) = C_1 u^{-2} \cos(\ln|u|) + C_2 u^{-2} \sin(\ln|u|)$

Plug back in

$\boxed{y(x) = C_1 (x-2)^{-2} \cos(\ln|x-2|) + C_2 (x-2)^{-2} \sin(\ln|x-2|)}$

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9. Find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x)^2y'' + 2xy' + 4y = 0.$$

singular pts are $x=0$ and $x=1$.

$x=0$:

$$\lim_{x\to 0} x \cdot \frac{2x}{x^2(1-x)^2} = \lim_{x\to 0} \frac{2}{(1-x)^2} = 2$$

$$\lim_{x\to 0} x^2 \frac{4}{x^2(1-x)^2} = \lim_{x\to 0} \frac{4}{(1-x)^2} = 4$$

so $x=0$ is a regular singular pt.

$x=1$:

$$\lim_{x\to 1} (x-1)\frac{2x}{x^2(1-x)^2} = \lim_{x\to 1} \frac{2}{x(1-x)} = \text{DNE}!$$

so $x=1$ is irregular.

10. Determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

(*a*.) $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$;

$P(x) = 1 - x^2 \qquad P(1/\xi) = 1 - \frac{1}{\xi^2}$

$Q(x) = -2x \qquad Q(1/\xi) = \frac{-2}{\xi}$

$R(x) = \alpha(\alpha+1) \qquad R(1/\xi) = \alpha(\alpha+1)$

$$p(\xi) = \frac{1}{\xi^2(1-\frac{1}{\xi^2})}\left(2\xi\left(1-\frac{1}{\xi^2}\right) + \frac{2}{\xi}\right) = \frac{1}{\xi^2-1}(2\xi) = \frac{2\xi}{\xi^2-1}$$

$$q(\xi) = \frac{\alpha(\alpha+1)}{\xi^4(1-1/\xi^2)} = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)}$$

x_∞ is singular because $q(\xi)$ is undefined at $\xi = 0$.

$$\xi p(\xi) = \frac{2\xi^2}{\xi^2-1}$$

$$\xi^2 q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2-1}$$

So x_∞ is regular b/c both of these are defined (have PS) at $\xi = 0$.

(*b*.) $y'' - 2xy' + \lambda y = 0$.

$P(x) = 1 \qquad P(1/\xi) = 1$

$Q(x) = -2x \qquad Q(1/\xi) = \frac{-2}{\xi}$

$R(x) = \lambda \qquad R(1/\xi) = \lambda$

$$p(\xi) = \frac{1}{\xi^2}\left(2\xi + \frac{2}{\xi}\right) = \frac{2(\xi^2+1)}{\xi^3}$$

$$q(\xi) = \frac{\lambda}{\xi^4}$$

So x_∞ is singular.

$$\xi p(\xi) = \frac{2(\xi^2+1)}{\xi^2}$$

$$\xi^2 q(\xi) = \frac{\lambda}{\xi^2}$$

so x_∞ is irregular.