Math 555: Differential Equations
Good Problems 4

Due: Friday, 27 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 8 problems. Each problem is worth 12 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:________________________

1. Consider the differential equation

$$y'' + 4y = t^2 + 3e^t.$$

(*a.*) Find the solution y_h of the corresponding homogeneous equation.

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$\boxed{y_h = C_1 \cos(2t) + C_2 \sin(2t)}$

(*b.*) Find the general solution $y = Y + y_h$ of the given differential equation.

Guess: $Y(t) = At^2 + Bt + C + De^t$

$Y' = 2At + B + De^t$

$Y'' = 2A + De^t$

Plugging in:

$Y'' + 4Y = 2A + De^t + 4At^2 + 4Bt + 4C + 4De^t = t^2 + 3e^t$

$\Rightarrow 4A\,t^2 + 4B\,t + (2A + 4C) + 5De^t = t^2 + 3e^t$

$\Rightarrow \begin{cases} 4A = 1 \\ 4B = 0 \\ 2A + 4C = 0 \\ 5D = 3 \end{cases} \Rightarrow \begin{array}{l} A = 1/4 \\ B = 0 \\ C = -\frac{1}{2}\left(\frac{1}{4}\right) = \frac{-1}{8} \\ D = 3/5 \end{array}$

So $Y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$ and

$\boxed{y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t + C_1 \cos(2t) + C_2 \sin(2t)}$

2. Consider the differential equation

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0.$$

(*a*.) Show that $y_1(t) = t$ is a solution of the DE.

$y_1 = t$

$y_1' = 1$

$y_1'' = 0$

So $t^2 y_1'' + 2ty_1' - 2y_1 = t^2(0) + 2t(1) - 2(t)$

$= 0 + 2t - 2t$

$= 0.$ □

(*b*.) Use the method of reduction of order to find a second solution of the DE.

Put $y = vt$ where $v = v(t)$ is an unknown function to be det.

$y' = tv' + v$

$y'' = tv'' + v' + v' = tv'' + 2v'$

Plugging in:

$t^2y'' + 2ty' - 2y = t^2(tv'' + 2v') + 2t(tv' + v) - 2(tv)$

$= t^3v'' + \underline{2t^2v' + 2t^2v'} + \cancel{2tv} - \cancel{2tv}$

$= t^3v'' + 4t^2v' = 0 \Rightarrow (v')' + \frac{4}{t}v' = 0$ (since $t \neq 0$)

So $\frac{dv'}{v'} = \frac{-4}{t}dt \Rightarrow \ln v' = -4\ln t + C_2$

$\Rightarrow v' = t^{-4}C_2$

$\Rightarrow v = -\frac{1}{3}C_2t^{-3} + C_1 = C_2t^{-3} + C_1$

So $y = tv = C_2t^{-2} + C_1t$ and $\boxed{y_2(t) = t^{-2}}$ or $\frac{1}{t^2}$

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3. Consider a differential equation of the form

$$t^2 y'' + \alpha t y' + \beta y = 0, \quad t < 0.$$

Making the substitution $t = -e^x$, repeat the calculation we did in class, *mutatis mutandis*, to transform the given equation into a SODE with constant coefficients.

To avoid notational issues, let $\dot{y} = \frac{dy}{dt}$ and $y' = \frac{dy}{dx}$.

Then the given DE says:

$$t^2 \ddot{y} + \alpha t \dot{y} + \beta y = 0 \quad , t < 0.$$

Put $t = -e^x$.

then $y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \cdot \underbrace{(-e^x)}_{=t} = \dot{y} \cdot t$ or $t\dot{y}$

and $y'' = (y')' = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\left[t\dot{y}\right] \cdot \frac{dt}{dx} = (\dot{y} + t\ddot{y}) \cdot \underbrace{(-e^x)}_{=t} = t^2\ddot{y} + t\dot{y}$

Then $\quad y'' + (\alpha - 1) y' + \beta y = t^2 \ddot{y} + t\dot{y} + (\alpha - 1) t\dot{y} + \beta y$

$$= t^2 \ddot{y} + \alpha t \dot{y} + \beta y = 0.$$

So the equation becomes

$$y'' + (\alpha - 1) y' + \beta y = 0$$

where $y = y(x)$! and $x = \ln|t|$.

4. Find the general solution of the differential equation

$$t^2 y'' + 3ty' + y = 0, \quad t < 0.$$

Euler $\longrightarrow$ $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

so $y(x) = C_1 e^{-x} + C_2 x e^{-x}$

subbing in $t = -e^{x}$ or $x = \ln(-t) = \ln|t|$,

$y(t) = C_1 e^{-\ln t} + C_2 \ln|t| e^{-\ln t}$

so

$$\boxed{y(t) = C_1 \frac{1}{t} + C_2 \frac{\ln|t|}{t}} \quad \underline{t<0}$$

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5. Solve the initial value problem

$$\begin{cases} y'' - 2y' + y = te^t + 4, \\ y(0) = 1, \\ y'(0) = 1. \end{cases}$$

homog: $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

$y_h = C_1 e^t + C_2 te^t$

because of y_h.

Guess: $Y(t) = \boxed{t^2}(At + B)e^t + C$

$= At^3e^t + Bt^2e^t + C$

$Y' = 3At^2e^t + At^3e^t + 2Bte^t + Bt^2e^t$

$= At^3e^t + (3A+B)t^2e^t + 2Bte^t$

$Y'' = 3At^2e^t + At^3e^t + 2(3A+B)te^t + (3A+B)t^2e^t + 2Be^t + 2Bte^t$

$= At^3e^t + (6A+B)t^2e^t + (6A+4B)te^t + 2Be^t$

Plug in: $Y'' - 2Y' + Y = \cancel{At^3e^t} + (\cancel{6A} + \cancel{B})t^2e^t + (6A + \cancel{4B})te^t + 2Be^t$

$-2\cancel{At^3e^t} - (\cancel{6A} + \cancel{2B})t^2e^t - \cancel{4B}te^t$

$+ \cancel{A}t^3e^t + \cancel{B}t^2e^t \qquad + C$

$0 + 0 + 6Ate^t + 4Be^t + C = te^t + 4$

Thus $6A = 1 \Rightarrow A = \frac{1}{6}$, $4B = 0$, $C = 4$ $\Big\} \Rightarrow Y(t) = \frac{1}{6}t^3e^t + 4$, and

$y(t) = \frac{1}{6}t^3e^t + 4 + C_1e^t + C_2te^t$

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$$y(t) = \frac{1}{6}t^3e^t + 4 + C_1e^t + C_2te^t$$

$$y'(t) = \frac{1}{2}t^2e^t + \frac{1}{6}t^3e^t + C_1e^t + C_2e^t + C_2te^t$$

$$y(0) = 4 + C_1 = 1 \Rightarrow C_1 = -3.$$

$$y'(0) = C_1 + C_2 = 1 \Rightarrow C_2 = 1 + 3 = 4$$

Thus, the final solution is

$$\boxed{y(t) = \frac{1}{6}t^3e^t + 4 - 3e^t + 4te^t}$$

6.

$$\frac{(1-t)y''}{1-t} + \frac{ty'}{1-t} - \frac{y}{1-t} = \frac{-2(t-1)^{2}e^{-t}}{\cancel{1-t}}$$

$$y_1 = e^t$$

$$y_2 = t$$

$$W(y_1, y_2) = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = (1-t)e^t$$

$$g = 2(1-t)e^{-t}$$

$$y = -e^t \int_{t_0}^{t} \frac{2s\cancel{(1-s)}e^{-s}}{\cancel{(1-s)}e^{s}}\,ds + t\int_{t_0}^{t} \frac{2e^{s}\cancel{(1-s)}e^{-s}}{\cancel{(1-s)}\cancel{e^{s}}}\,ds$$

$$= -2e^t \int_{t_0}^{t} s e^{-2s}\,ds + 2t\int_{t_0}^{t} e^{-s}\,ds$$

$$= -2e^t\left(-\frac{1}{2}se^{-2s} - \frac{1}{4}e^{-2s}\right)\Big|_{t_0}^{t} + 2t\left(-e^{-s}\right)\Big|_{t_0}^{t}$$

$$= -2e^t\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c_1\right) + 2t\left(-e^{-t} + c_2\right)$$

$$= te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} + c_1e^t + c_2t$$

$$\boxed{y = -te^{-t} + \frac{1}{2}e^{-t} + c_1e^t + c_2t}$$

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7. Find the general solution of the differential equation

$$\begin{cases} y'' + 9y = 9\sec^2(3t), \\ 0 < t < \pi/6. \end{cases}$$

homog: $y'' + 9y = 0$, $r^2 + 9 = 0$, $r = \pm 3i$ $\Bigg\}$ $y_h = C_1 \cos(3t) + C_2 \sin(3t)$, $y_1 = \cos(3t)$, $y_2 = \sin(3t)$

$$W(y_1, y_2) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3.$$

$g(t) = 9\sec^2(3t)$

$$y(t) = -\cos(3t)\int_{t_0}^{t} \frac{\sin(3s)\, 9\sec^2(3s)}{3}\, ds + \sin(3t)\int_{t_0}^{t} \frac{\cos(3s)\, 9\sec^2(3s)}{3}\, ds$$

$$= -3\cos(3t)\underbrace{\int_{t_0}^{t} \frac{\sin(3s)}{\cos^2(3s)}\, ds}_{I_1} + 3\sin(3t)\underbrace{\int_{t_0}^{t} \sec(3s)\, ds}_{I_2}$$

$$I_1 = \int_{t_0}^{t} \frac{\sin(3s)}{\cos^2(3s)}\, ds \quad \left\{ \begin{array}{l} u = \cos(3s) \\ du = -3\sin(3s)\, ds \end{array} \right\} = -\frac{1}{3}\int_{t_0}^{\cos(3t)} \frac{1}{u^2}\, du = \frac{1}{3}\cdot\frac{1}{u}\Bigg|_{t_0}^{\cos(3t)} = \frac{1}{3}\sec(3t) + C_1$$

$$I_2 = \int_{t_0}^{t} \sec(3s)\, ds \quad \left\{ \begin{array}{l} u = 3s \\ du = 3ds \end{array} \right\} = \frac{1}{3}\int_{t_0}^{3t} \sec u\, du = \frac{1}{3}\ln|\sec u + \tan u|\Bigg|_{t_0}^{3t}$$

$$= \frac{1}{3}\ln|\sec(3t) + \tan(3t)| + C_2$$

so, $y(t) = -3\cos(3t)\left[\frac{1}{3}\sec(3t) + C_1\right] + 3\sin(3t)\left[\frac{1}{3}\ln|\sec(3t) + \tan(3t)| + C_2\right]$

$$\boxed{y(t) = -1 + C_1\cos(3t) + \sin(3t)\ln|\sec(3t) + \tan(3t)| + C_2\sin(3t)}$$

8.

$$t^2y'' - 3ty' + 4y = t^2\ln t, \quad t > 0.$$

homog: $t^2 y'' - 3ty' + 4y = 0$

$$r^2 - 4r + 4 = 0$$

$$r = 2, 2$$

$$y_1 = t^2$$

$$y_2 = t^2\ln t$$

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^2\ln t \\ 2t & 2t\ln t + t \end{vmatrix} = 2t^3\ln t + t^3 - 2t^3\ln t = t^3.$$

$$g = \frac{t^2\ln t}{t^2} = \ln t$$

$$y = -t^2\int_{t_0}^{t} \frac{s^2\ln s \cdot \ln s}{s^3}\,ds + t^2\ln t\int_{t_0}^{t}\frac{s^2 \cdot \ln s}{s^3}\,ds$$

$$= -t^2\int_{t_0}^{t}\frac{(\ln s)^2}{s}\,ds + t^2\ln t\int_{t_0}^{t}\frac{\ln s}{s}\,ds$$

$$\left\{\begin{matrix} u = \ln s \\ du = \frac{1}{s}ds \end{matrix}\right\}$$

$$= -t^2\int_{u_0}^{\ln t} u^2\,du + t^2\ln t\int_{u_0}^{\ln t} u\,du$$

$$= -t^2\left(\frac{1}{3}u^3\right)\Big|_{u_0}^{\ln t} + t^2\ln t\left(\frac{1}{2}u^2\right)_{u_0}^{\ln t}$$

$$y = -\frac{1}{3}t^2(\ln t)^3 + \frac{1}{2}t^2(\ln t)^3 + C_1t^2 + C_2t^2\ln t$$

$$\boxed{y = \frac{1}{6}t^2(\ln t)^3 + C_1t^2 + C_2t^2\ln t}$$