Math 555: Differential Equations
Good Problems 3

Due: Friday, 20 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 8 problems. Each problem is worth 12 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name: ____________________________________

1–3. Find the general solution of the differential equation.

1. $6y'' - y' - y = 0$

$6r^2 - r - 1 = 0$

$6r^2 - 3r + 2r - 1 = 0$

$3r(2r-1) + (2r-1) = 0$

$(3r+1)(2r-1) = 0$

$r_1 = \frac{-1}{3} \quad r_2 = \frac{1}{2}$

$$\boxed{y = C_1 e^{-\frac{1}{3}t} + C_2 e^{\frac{1}{2}t}}$$

2. $y'' - 2y' + 6y = 0$

$r^2 - 2r + 6 = 0$

$r^2 - 2r + 1 = -5$

$(r-1)^2 = -5$

$r = 1 \pm \sqrt{-5}$

$r = 1 \pm i\sqrt{5}$

$$\boxed{y = e^{t}\left(C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t)\right)}$$

3. $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$$\boxed{y = C_1 e^{t} + C_2 t e^{t}}$$

4–5. Find the particular solution of the initial value problem.

4. $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$

$y(0) = C_1 = 1$

$y'(0) = -2C_1 + C_2 = 0$

so $C_2 = 2$.

$r^2 + 4r + 5 = 0$

$(r+2)^2 + 1 = 0$

$r = -2 \pm i$

$\boxed{y(t) = e^{-2t}\cos t + 2e^{-2t}\sin t}$

$y = e^{-2t}(C_1 \cos t + C_2 \sin t)$

$y = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$

$y' = -2C_1 e^{-2t}\cos t - C_1 e^{-2t}\sin t - 2C_2 e^{-2t}\sin t + C_2 e^{-2t}\cos t$

5. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = -1. \end{cases}$

$r^2 + 4r + 3 = 0$

$(r+2)^2 = 1$

$r = -2 \pm 1$

$r_1 = -1, \quad r_2 = -3$

$\boxed{y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}}$

$y(t) = C_1 e^{-t} + C_2 e^{-3t}$

$y'(t) = -C_1 e^{-t} - 3C_2 e^{-3t}$

$y(0) = C_1 + C_2 = 2$

$y'(0) = -C_1 - 3C_2 = -1$

$-2C_2 = 1$

$C_2 = \frac{-1}{2}$

$C_1 = \frac{5}{2}$

Name:______________________________

6–7. Find the Wronskian of the set of functions.

6. $\{e^{2t}\cos(t), e^{2t}\sin(t)\}$

$$W[e^{2t}\cos(t), e^{2t}\sin(t)] = \begin{vmatrix} e^{2t}\cos t & e^{2t}\sin t \\ 2e^{2t}\cos t - e^{2t}\sin t & 2e^{2t}\sin t + e^{2t}\cos t \end{vmatrix}$$

$$= \cancel{2e^{4t}\sin t\cos t} + e^{4t}\cos^2 t \; \cancel{- 2e^{4t}\cos t\sin t} + e^{4t}\sin^2 t$$

$$= e^{4t}(\cos^2 t + \sin^2 t) = \boxed{e^{4t}} \neq 0$$

7. $\{e^t, te^t\}$

$$W[e^t, te^t] = \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = \boxed{e^{2t}} \neq 0.$$

8. Suppose $\varphi_1(x) = e^x$ is a solution of the differential equation

$$(x-1)y'' - xy' + y = 0, \quad x > 0.$$

Use the method of reduction of order to find a second solution of the differential equation. Verify that you have indeed found a *new* solution.

$\varphi_1(x) = e^x$

Put $y = v(x)e^x$

then $y' = e^x(v(x) + v'(x))$

and $y'' = e^x(v(x) + 2v'(x) + v''(x))$

Plugging into the DE:

$$(x-1)y'' - xy' + y = (x-1)e^x(v + 2v' + v'') - xe^x(v + v') + e^x v = 0$$

$e^x \neq 0$ can be cancelled. Distribute the rest:

$$v''(x-1) + v'(2(x-1) - x) + v(\overset{0}{\cancel{(x-1) - x + 1}}) = 0$$

$$(x-1)v'' + (x-2)v' = 0 \qquad \text{this is a separable FODE for } v'!$$

$$(x-1)\frac{dv'}{dx} = (2-x)v'$$

$$\frac{dv'}{v'} = \frac{2-x}{x-1}\,dx$$

$$\ln(v') = \int -\frac{(x-1)}{x-1} + \frac{1}{x-1}\,dx = -x + \ln(x-1) + c$$

so $v' = C_2(x-1)e^{-x}$

$$\frac{dv}{dx} = C_2 xe^{-x} - C_2 e^{-x}$$

$$v(x) = \int C_2 xe^{-x}\,dx - \int C_2 e^{-x}\,dx = C_2(-xe^{-x} \cancel{- e^{-x} + e^{-x}}) + C_1 = \boxed{C_2 xe^{-x} + C_1 = v}$$

Thus

$$y = ve^x = \underbrace{C_1e^x}_{\text{old}} + \underbrace{C_2 x}_{\text{new}}$$

and $\boxed{y_2(x) = x.}$

$W[x, e^x] = xe^x - e^x \neq 0$ if $x \neq 1$.