Math 555: Differential Equations
Good Problems 2

Due: Friday, 13 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: KEY

Instructions: Complete all 8 problems. Each problem is worth 12 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:____________________________________

1. Find the general solution of the differential equation

$$y' + \tfrac{1}{t}y = t^2\sin(t).$$

$p(t) = \frac{1}{t}$

$g(t) = t^2 \sin(t)$

$\mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$y(t) = \frac{1}{t}\int_{t_0}^{t} s^3 \sin(s)\,ds + \frac{C}{t}$

Integrate by parts: $\int s^3 \sin(s)\,ds$

	u	dv
+	s^3	$\sin(s)$
−	$3s^2$	$-\cos(s)$
+	$6s$	$-\sin(s)$
−	6	$\cos(s)$
	0	$\sin(s)$

$= -s^3\cos(s) + 3s^2\sin(s) + 6s\cos(s) - 6\sin(s)$

So, $y(t) = \frac{1}{t}\left(-t^3\cos(t) + 3t^2\sin(t) + 6t\cos(t) - 6\sin(t)\right) + \frac{C}{t}$

or $\boxed{y(t) = -t^2\cos(t) + 3t\sin(t) + 6\cos(t) - \frac{6}{t}\sin(t) + \frac{C}{t}}$

2. Solve the initial value problem

$$\begin{cases} y' + 2ty = 4t^3, \\ y(0) = 1. \end{cases}$$

$p(t) = 2t$

$g(t) = 4t^3$

$\mu(t) = e^{\int 2t\,dt} = e^{t^2}$

$y(t) = e^{-t^2} \int_{t_0}^{t} 4s^3 e^{s^2}\,ds + y_0 e^{-t^2}$

$= e^{-t^2} \int_0^t 4s^3 e^{s^2}\,ds + e^{-t^2}$

$u = s^2 \qquad du = 2s\,ds$

$u(t) = t^2$

$u(0) = 0^2 = 0$

$= e^{-t^2} \int_0^{t^2} 2u e^u\,du + e^{-t^2}$

$= e^{-t^2} \left(2ue^u - 2e^u \right) \Big|_0^{t^2} + e^{-t^2}$

$= e^{-t^2} \left(2t^2 e^{t^2} - 2e^{t^2} - (0-2) \right) + e^{-t^2}$

$= 2t^2 - 2 + 2e^{-t^2} + e^{-t^2}$

$\boxed{y(t) = 2t^2 - 2 + 3e^{-t^2}}$

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3. Find the general solution of the differential equation

$$\frac{dy}{dx} = y' = \frac{x}{4y + 4x^2 y}.$$

$$\int 4y \, dy = \frac{1}{2}\int \frac{2x}{1+x^2} \, dx$$

$u = 1+x^2$

$du = 2x \, dx$

$$\boxed{2y^2 = \frac{1}{2}\ln(1+x^2) + C}$$

4. 1. Prove that every separable equation $M(x) + N(y)y' = 0$ is exact.

2. Solve the initial value problem

$$(2x - y) - (x - 2y)y' = 0; \quad y(1) = 3.$$

1.) $\left.\begin{aligned} &\partial_y M(x) = 0 \\ &\partial_x N(y) = 0 \end{aligned}\right\}$

Since these are equal, the DE is exact.

2.) $\left.\begin{aligned} &\partial_y (2x-y) = -1 \\ &\partial_x (-x+2y) = -1 \end{aligned}\right\}$ exact!

$$\psi(x,y) = \int_1^x 2s-3\,ds + \int_3^y 2t-1\,dt = 0$$

$$(s^2-3s)\Big|_1^x + (t^2-t)\Big|_3^y = 0$$

$$x^2-3x-(1-3) + y^2-y-(9-3) = 0$$

$$y^2 - y + (x^2-3x-4) = 0$$

$$y = \frac{+1 \pm \sqrt{1-4(x^2-3x-4)}}{2}$$

$$y(1) = \frac{1 \pm \sqrt{1-4(1-3-4)}}{2} = \frac{1 \pm 5}{2} = \overset{+}{3}, \overset{-}{-2}$$

so take the + sign:

$$\boxed{y(t) = \frac{1 + \sqrt{1-4(x^2-3x-4)}}{2}}$$

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5. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}, \\ y(0) = 3. \end{cases}$$

$$2y-2\,dy - (3x^2+4x+2)\,dx = 0$$

$$\int_3^y 2s-2\,ds - \int_0^x 3t^2+4t+2\,dt = 0$$

$$(s^2-2s)\Big|_3^y - (t^3+2t^2+2t)\Big|_0^x = 0$$

$$y^2-2y-(9-6)-\left[x^3+2x^2+2x-0\right]=0$$

$$y^2-2y+\left(-x^3-2x^2-2x-3\right)=0$$

$$y = \frac{2\pm\sqrt{4-4(-3-x^3-2x^2-2x)}}{2} = 1\pm\sqrt{1-(-3-x^3-2x^2-2x)}$$

or $\quad y = 1\pm\sqrt{x^3+2x^2+2x+4}$

$$y(0) = 1\pm 2 = 3,-1$$

so $\quad \boxed{y(x) = 1+\sqrt{x^3+2x^2+2x+4}}$

6. Consider the differential equation

$$\left(x^2 y + y\sin(x) - 2x\right)dx + \left(\frac{2}{3}x^3 - 2\cos(x) - \frac{x^2}{y}\right)dy = 0.$$

1. Show that this DE is *not* exact.
2. Multiply through the equation by the integrating factor $\mu(x, y) = y$.
3. Verify that the DE you obtain from the last step *is* exact.
4. Find the solution of the original DE.

1.)

$$\left.\begin{aligned}\partial_y\left(x^2y + y\sin(x) - 2x\right) &= x^2 + \sin(x)\\ \partial_x\left(\tfrac{2}{3}x^3 - 2\cos(x) - \tfrac{x^2}{y}\right) &= 2x^2 + 2\sin(x) - 2\tfrac{x}{y}\end{aligned}\right\} \neq .$$

2.)

$$\mu M\,dx + \mu N\,dy = \left(x^2y^2 + y^2\sin(x) - 2xy\right)dx + \left(\tfrac{2}{3}x^3y - 2y\cos x - x^2\right)dy = 0$$

3.)

$$\left.\begin{aligned}\partial_y\left(x^2y^2 + y^2\sin x - 2xy\right) &= 2x^2y + 2y\sin x - 2x\\ \partial_x\left(\tfrac{2}{3}x^3y - 2y\cos x - x^2\right) &= 2x^2y + 2y\sin x - 2x\end{aligned}\right\} \text{exact!}$$

4.)

$$\psi(x,y) = \int x^2y^2 + y^2\sin x - 2xy\,dx = \tfrac{1}{3}x^3y^2 - y^2\cos x - x^2y + f(y)$$

$$\psi(x,y) = \int \tfrac{2}{3}x^3y - 2y\cos x - x^2\,dy = \tfrac{1}{3}x^3y^2 - y^2\cos x - x^2y + h(x)$$

soln: $\boxed{\psi(x,y) = \tfrac{1}{3}x^3y^2 - y^2\cos x - x^2y = C}$

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7. Consider the initial value problem (IVP):

$$\begin{cases} t(t-4)y' + y = 0, \\ y(2) = 1. \end{cases}$$

1. First, find the domain of definition of the solution <u>without</u> solving the differential equation.
2. Now, solve the IVP explicitly and then find the domain of the solution.

1.) $y' + \frac{1}{t(t-4)} y = 0.$

$p(t) = \frac{1}{t(t-4)}$ $\quad$ $\mathrm{dom}(p) = (-\infty, 0) \cup (0,4) \cup (4, \infty)$

$g(t) = 0$ $\quad$ $\mathrm{dom}(g) = (-\infty, \infty)$

$t_0 = 2$, so $\boxed{\mathrm{dom}(y) = (0,4)}$

2.) $\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t(t-4)}\,dt} = e^{-\frac{1}{4}\int \frac{1}{t}\,dt + \frac{1}{4}\int \frac{1}{t-4}\,dt} = e^{-\frac{1}{4}\ln t}\, e^{\frac{1}{4}\ln|t-4|}$

$= t^{-\frac{1}{4}} \cdot |t-4|^{1/4}$

$= \left(\frac{|t-4|}{t}\right)^{1/4}$

$\frac{1}{t(t-4)} = \frac{A}{t} + \frac{B}{t-4} \rightarrow A(t-4) + Bt = 1$

$t = 0: \Rightarrow -4A = 1 \Rightarrow A = -\frac{1}{4}$

$t = 4: \Rightarrow 4B = 1 \Rightarrow B = \frac{1}{4}$

$\boxed{y(t) = C\sqrt[4]{\frac{t}{|t-4|}}} = C\left(\frac{t}{|t-4|}\right)^{1/4}$

$y(2) = C\left(\frac{2}{2}\right)^{1/4} = C = 1$

so $y(t) = \sqrt[4]{\frac{t}{4-t}}$

$\mathrm{dom}(y) = (0,4).$

8. Consider the initial value problem

$$\begin{cases} y' = ty + t^2, \\ y(0) = 1. \end{cases}$$

1. Write down the general form of the Picard operator $\mathcal{P}f$ for this problem.
2. Let $\varphi_0 = 1$. Calculate φ_1, φ_2, φ_3, and φ_4 using Picard's iterative method.
3. (Don't turn this part in.) Graph the functions $\varphi_0, \ldots, \varphi_4$ on the same set of axes. Compare their graphs.

1.) $\mathcal{P}f(y) = \int_0^t sy + s^2 \, ds + 1$

2. $\varphi_1(t) = \mathcal{P}f(\varphi_0)(t) = \int_0^t s + s^2 \, ds + 1 = \left(\frac{1}{2}s^2 + \frac{1}{3}s^3\right)_0^t + 1$

$= \boxed{\frac{1}{2}t^2 + \frac{1}{3}t^3 + 1 = \varphi_1}$

$\varphi_2(t) = \mathcal{P}f(\varphi_1)(t) = \int_0^t s\left(\frac{1}{2}s^2 + \frac{1}{3}s^3 + 1\right) + s^2 \, ds + 1$

$= \int_0^t \frac{1}{3}s^4 + \frac{1}{2}s^3 + s^2 + s \, ds + 1$

$= \boxed{\frac{1}{15}t^5 + \frac{1}{8}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 + 1 = \varphi_2}$

$\varphi_3(t) = \mathcal{P}f(\varphi_2)(t) = \int_0^t s\left(\frac{1}{15}s^5 + \frac{1}{8}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + 1\right) + s^2 \, ds + 1$

$= \int_0^t \frac{1}{15}s^6 + \frac{1}{8}s^5 + \frac{1}{3}s^4 + \frac{1}{2}s^3 + s^2 + s \, ds + 1$

$= \boxed{\frac{1}{105}t^7 + \frac{1}{48}t^6 + \frac{1}{15}t^5 + \frac{1}{8}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 + 1 = \varphi_3}$

Similarly,

$\varphi_4(t) =$