Math 555: Differential Equations
Good Problems 1

Due: Friday, 6 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: KEY

Instructions: Complete all 8 problems. Each problem is worth 12 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

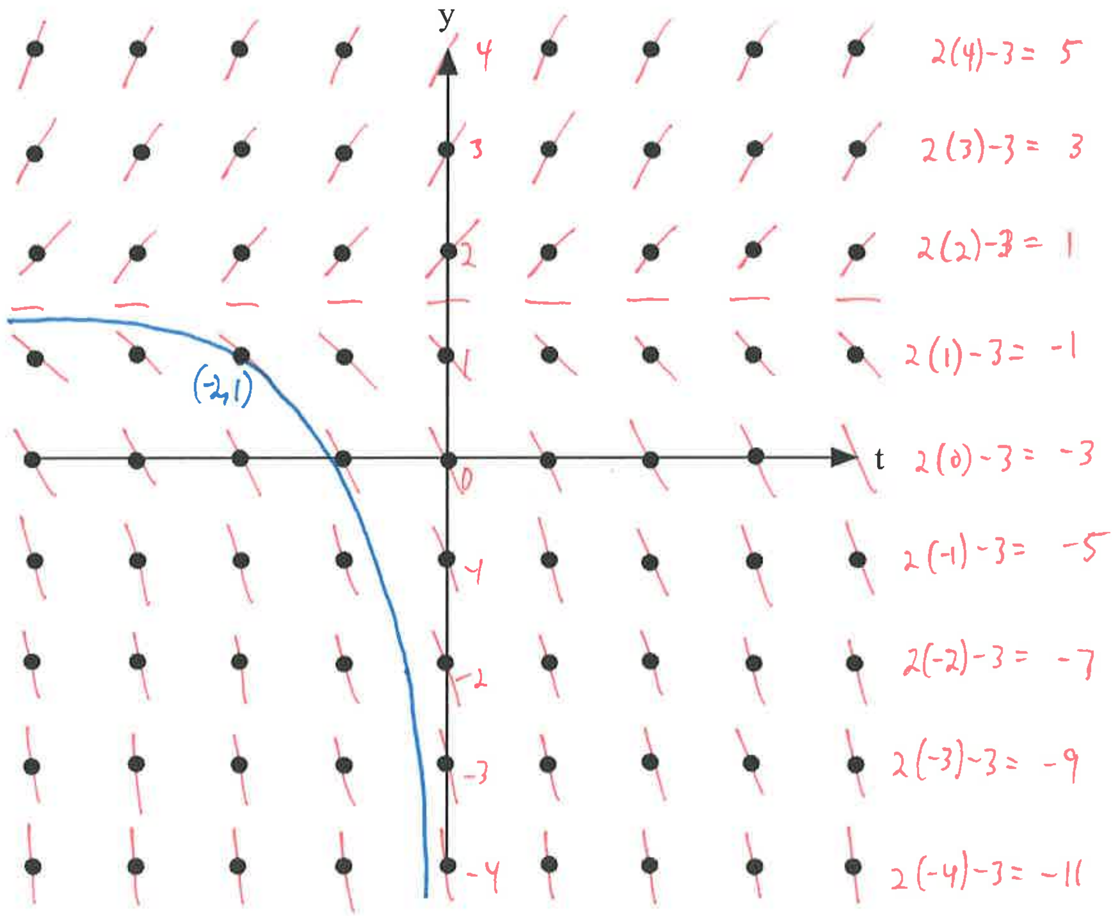
Good Luck!

Name:______________________________

1. Draw a direction field for the differential equation

$$y' = 2y - 3$$

on the dot paper provided.



Sketch the integral curve of the DE with initial condition $y(-2) = 1$.

blue curve is the integral curve.

What is the equilibrium solution of this DE?

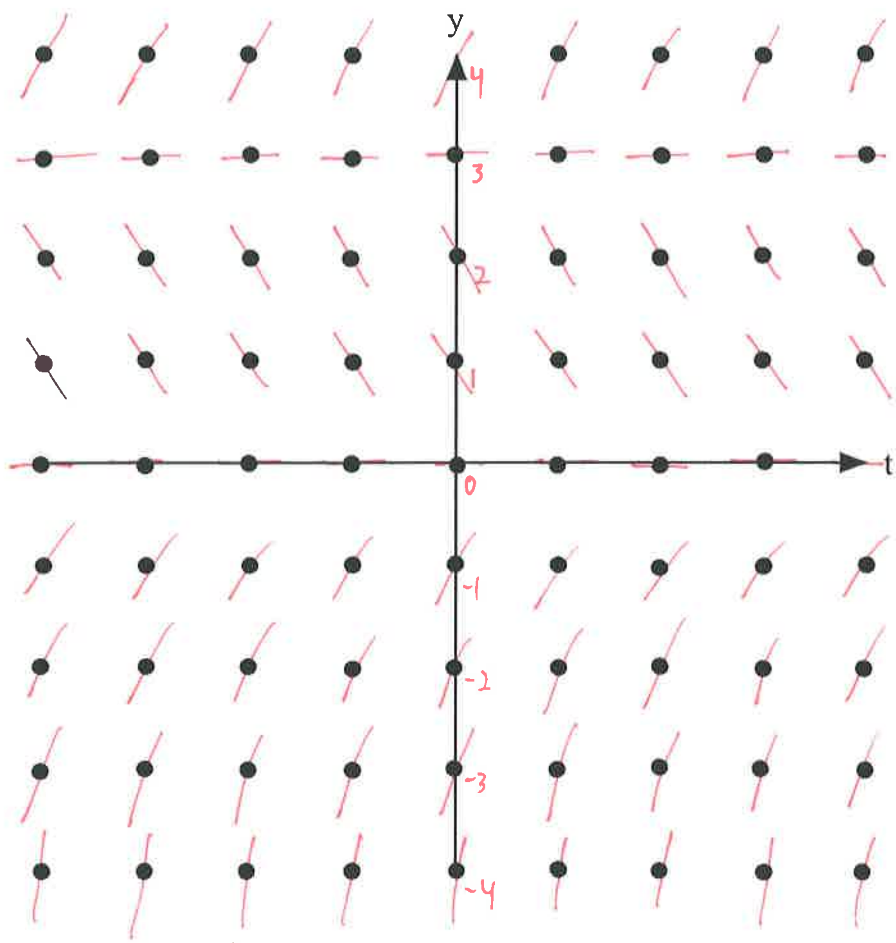
$0 = 2y - 3$

$\Rightarrow y_e = \frac{3}{2}$

2. Sketch a direction field for the differential equation

$$y' = y(y-3)$$

on the dot paper provided.



$4(4-3) = 4$

$3(3-3) = 0$

$2(2-3) = -2$

$1(1-2) = -2$

$0(0-3) = 0$

$-1(-1-3) = 4$

$-2(-2-3) = 10$

$-3(-3-3) = 18$

$-4(-4-3) = 28$

Name:______________________________

3. Consider the differential equation $y' = ay - b$.

1. Find the equilibrium solution φ_e.

2. For any other solution φ of the differential equation, put $Y(t) = \varphi(t) - \varphi_e(t)$. Thus Y measures the deviation of φ from the equilibrium solution. Find the differential equation satisfied by Y.

1.) $0 = ay - b \Rightarrow y = \frac{b}{a}$ so $\boxed{\varphi_e = \frac{b}{a}}$ constant function.

2.) let φ be a solution; i.e., $\varphi' = a\varphi - b$.

Put $Y = \varphi - \varphi_e = \varphi - \frac{b}{a}$ so $\varphi = Y + \frac{b}{a}$

Then $Y' = \varphi'$

But $\varphi' = a\varphi - b = a\left(Y + \frac{b}{a}\right) - b = aY + b - b = aY$

Thus Y solves the DE: $\boxed{Y' = aY}$.

4. Consider the differential equation $y' = -2y + 5$.

1. First solve the differential equation $y' = -2y$. Call the solution $y = \varphi_1(t)$.
2. Put $Y(t) = \varphi_1(t) + k$, where k is an unknown constant.
3. Compute $Y'(t)$.
4. Plug these functions into the equation $Y' = -2Y + 5$, and solve for k.
5. What is the solution of the original differential equation?

1.) $y' = -2y \Rightarrow \int \frac{dy}{y} = \int -2\, dt \Rightarrow \ln y = -2t + C$

$\Rightarrow y = Ce^{-2t}$

$\Rightarrow \boxed{\varphi_1(t) = Ce^{-2t}}$

2.) $Y(t) = Ce^{-2t} + k$

3.) $Y'(t) = -2Ce^{-2t}$

4.) $Y'(t) = -2Y + 5$

$-2Ce^{-2t} = -2(Ce^{-2t} + k) + 5$

$\Rightarrow \cancel{-2Ce^{-2t}} = \cancel{-2Ce^{-2t}} + 2k + 5$

$\Rightarrow 0 = 2k + 5$

$\Rightarrow \boxed{k = \frac{-5}{2}}$

5.) Thus $\boxed{Y(t) = Ce^{-2t} - \frac{5}{2}}$ solve the original DE: $y' = -2y + 5$.

Name:______________________________

5. The *half-life* of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $y' = -ry$, $r > 0$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln(2)$.

First solve the IVP $y' = -ry$, $y(0) = y_0$.

Solution is: $y(t) = y_0 e^{-rt}$ [we've solved similar problems many times... see problem 4, for example.]

Now half-life means that $y(\tau) = \frac{1}{2} y_0$.

But $y(\tau) = y_0 e^{-r\tau}$

Setting these equal and solving for $r\tau$,

$$\frac{1}{2}\cancel{y_0} = \cancel{y_0} e^{-r\tau}$$

$$\frac{1}{2} = e^{-r\tau}$$

$$\ln\left(\frac{1}{2}\right) = -r\tau$$

$$\Rightarrow \quad r\tau = -\ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad r\tau = \ln(2). \quad \square$$

6. Verify that the function

$$\varphi(t) = e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}$$

is a solution of the differential equation $y' - 2ty = 1$.

We must show that $\varphi' - 2t\varphi = 1$.

$$\varphi' = \frac{d}{dt}\left[e^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}\right] = \frac{d}{dt}\left[e^{t^2}\int_0^t e^{-s^2}ds\right] + \frac{d}{dt}\left[e^{t^2}\right]$$

$$= \frac{d}{dt}\left[e^{t^2}\right]\int_0^t e^{-s^2}ds + e^{t^2}\frac{d}{dt}\left[\int_0^t e^{-s^2}ds\right] + \frac{d}{dt}\left[e^{t^2}\right]$$

$$= 2te^{t^2}\int_0^t e^{-s^2}ds + \underbrace{e^{t^2}e^{-t^2}}_{=1} + 2te^{t^2}$$

Plugging into the DE:

$$\varphi' - 2t\varphi = \underline{2te^{t^2}\int_0^t e^{-s^2}ds} + 1 + 2te^{t^2} - \underline{2te^{t^2}\int_0^t e^{-s^2}ds} - 2te^{t^2}$$

$$= 1. \quad \square$$

Name:________________________________

7. Find all values of r such that $\varphi(t) = e^{rt}$ is a solution of the differential equation

$$y''' - 3y'' + 2y' = 0.$$

$$\left.\begin{aligned} \varphi' &= re^{rt} \\ \varphi'' &= r^2 e^{rt} \\ \varphi''' &= r^3 e^{rt} \end{aligned}\right\}$$ Plug into DE:

$$r^3 e^{rt} - 3r^2 e^{rt} + 2re^{rt} = 0$$

$$re^{rt}\left(r^2 - 3r + 2\right) = 0$$

$$e^{rt} \cdot r \cdot (r-2) \cdot (r-1) = 0$$

So $\boxed{r = 0, 1, 2}$

So solutions of the DE are

$$\left\{\begin{aligned} \varphi_1 &= e^{0t} = 1 \\ \varphi_2 &= e^{t} \\ \varphi_3 &= e^{2t}. \end{aligned}\right.$$

8. Solve the equation

$$\frac{dy}{dx} = \frac{ay+b}{cy+d},$$

where a, b, c, d are constants.

Use this general formula to write down the solution of the equation $\dfrac{dy}{dx} = \dfrac{y}{y+2}$.

$$\frac{dy}{dx} = \frac{a}{c}\left(\frac{y + b/a}{y + d/c}\right)$$

$$y + b/a \,\overline{)\, y \,}\quad 1 - \frac{b/a}{y+b/a}$$
$$\frac{y + b/a}{- b/a}$$

$$\left(\frac{y + d/c}{y + b/a}\right) dy = \frac{a}{c}\, dx$$

$$\left(\frac{y}{y + b/a} + \frac{d/c}{y + b/a}\right) dy = \frac{a}{c}\, dx$$

$$\Rightarrow \int\left(1 + \frac{d/c - b/a}{y + b/a}\right) dy = \int \frac{a}{c}\, dx$$

$$\boxed{y + \frac{ad - bc}{ac} \ln\left(y + b/a\right) = \frac{a}{c}\, x + C}$$

This is the best we can hope for in general, I think.

In the DE $\frac{dy}{dx} = \frac{y}{y+2}$ $a=1,\ b=0,\ c=1,\ d=2$

so the general solution becomes:

$$y + \frac{1\cdot 2 - 0\cdot 1}{1\cdot 1} \ln\left(y + \frac{0}{1}\right) = \frac{1}{1}\, x + C \qquad \text{or}$$

$$\boxed{y + 2\ln(y) = x + C}$$