Math 555: Differential Equations
Final Exam, Part II

Friday, 25 July 2014

Name: Key

Instructions: Complete 5 of the 6 problems. Clearly mark the problem that you would like to omit. Each completed problem is worth 20 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use three 3×5 index cards of your own notes (or one side of a page of notebook paper), a pencil, and your brain.

Good Luck!

Name: ____________________

Instructions. Complete 5 of 6 problems in the space provided. Show enough work. Clearly mark the one problem that you wish to omit.

1. Solve the initial value problem

$$\begin{cases} ty' + 3y = \dfrac{\sin t}{t^2}; \\ y(0) = -2. \end{cases}$$

First-order linear

$$y' + \frac{3}{t} y = \frac{\sin t}{t^3}$$

$$\mu(t) = e^{\int 3/t \, dt} = e^{3\ln t} = t^3$$

$$y(t) = \frac{1}{t^3} \int_0^t \cancel{s^3} \frac{\sin(s)}{\cancel{s^3}} \, ds - \frac{2}{t^3}$$

$$= \frac{1}{t^3} \left[-\cos t + \cos 0 \right] - \frac{2}{t^3}$$

$$= \frac{-\cos t}{t^3} - \frac{1}{t^3} = \boxed{\frac{-(\cos t + 1)}{t^3} = y}$$

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2. Find the general solution of the DE.

$$y'' - 2y' + 2y = te^t + 2$$

Second-order linear constant coeff. non-homog.

homog: $y''-2y'+2y=0$

$r^2-2r+2=0$

$(r-1)^2+1=0$

$r=1\pm i$

$y_h = C_1e^t\cos t + C_2e^t\sin t$

non-homog: $Y(t)= Ate^t + Be^t + C$

$Y'(t) = Ate^t + (A+B)e^t$

$Y''(t) = Ate^t + (2A+B)e^t$

$Y''-2Y'+2Y = Ate^t + (\cancel{2A}+B)e^t - 2\cancel{Ate^t} - 2(\cancel{A+B})e^t + \cancel{2Ate^t} + \cancel{2Be^t} + 2C = te^t + 2$

$\Rightarrow A=1$, $B=0$, $C=1$ $\Rightarrow Y(t) = te^t + 1$

Thus, $\boxed{y = C_1e^t\cos t + C_2e^t\sin t + te^t + 1}$

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3. Solve the initial value problem

$$\begin{cases} t^2 y'' + t y' - y = t^2 \ln t; \\ y(1) = 0. \end{cases}$$

2nd-order linear

$$y'' + \frac{1}{t} y' - \frac{1}{t^2} y = \ln t$$

homog: $t^2 y'' + t y' - y = 0$ Euler!

$r^2 - 1 = 0$

$r = \pm 1$

$y_1(t) = t$

$y_2(t) = 1/t$

Variation of Parameters:

$$W(t, 1/t) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = \frac{-2}{t} \qquad g(t) = \ln t$$

$$y = -t \int_1^t \frac{\frac{1}{s} \ln(s)}{-2/s}\, ds + \frac{1}{t} \int_1^t \frac{s \ln s}{-2/s}\, ds$$

$$= \frac{t}{2} \int_1^t \ln s \, ds - \frac{1}{2}\frac{1}{t} \underbrace{\int_1^t s^2 \ln s \, ds}$$

$u = \ln s$ $\quad dv = s^2$

$du = \frac{1}{s} ds$ $\quad v = \frac{1}{3} s^3$

IbP: $= \frac{1}{3} s^3 \ln s - \frac{1}{9} s^3$

$$= \frac{t}{2} \left[s \ln s - s \Big|_1^t \right] - \frac{1}{2}\frac{1}{t} \left[\frac{1}{3} s^3 \ln s - \frac{1}{9} s^3 \right]$$

$$= \frac{t}{2} \left(t \ln t - t + 1 \right) - \frac{1}{2}\frac{1}{t} \left[\frac{1}{3} t^3 \ln t - \frac{1}{9} t^3 + \frac{1}{9} \right]$$

$$= \frac{1}{2} t^2 \ln t - \frac{1}{2} t^2 + \frac{t}{2} - \frac{1}{6} t^2 \ln t + \frac{1}{18} t^2 - \frac{1}{18}\frac{1}{t}$$

$$\boxed{y = \frac{1}{3} t^2 \ln t - \frac{4}{9} t^2 + \frac{1}{2} t - \frac{1}{18}\frac{1}{t}}$$

$$y(1) = 0 - \frac{4}{9} + \frac{1}{2} - \frac{1}{18} = \frac{-8 + 9 - 1}{18} = 0. \checkmark$$

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4. Given that $y_1(x) = x^3$ is a solution of the differential equation

$$x^2y'' - 5xy' + 9y = 0,$$

use the method of reduction of order to find a second solution. Clearly label the new solution y_2.

$y_1(x) = x^3$

Put $y(x) = v(x)x^3 = vx^3$

$y' = v'x^3 + 3vx^2$

$y'' = v''x^3 + 3v'x^2 + 3v'x^2 + 6vx = v''x^3 + 6v'x^2 + 6vx$

Plug in:

$x^2y'' - 5xy' + 9y = x^2(v''x^3 + 6v'x^2 + 6vx) - 5x(v'x^3 + 3vx^2) + 9vx^3 = 0$

$\Rightarrow v''x^5 + \overbrace{[6v'x^4 - 5v'x^4]}^{v'x^4} + \overbrace{[6vx^3 \cancel{- 15vx^3 + 9vx^3}]}^{0} = 0$

$\Rightarrow \quad x^4(v''x + v') = 0$

$\Rightarrow v''x + v' = 0$

$\Rightarrow (v')' = -\frac{v'}{x} \Rightarrow \frac{dv'}{v'} = -\frac{1}{x}\,dx \Rightarrow \ln(v') = -\ln x + C_1$

$\Rightarrow v' = C_1 \cdot \frac{1}{x}$

$\Rightarrow v = C_1 \ln x + C_2$

So $y = vx^3 = \underbrace{C_1 x^3 \ln x}_{\text{new}} + \underbrace{C_2 x^3}_{\text{old}}$

Thus $\boxed{y_2 = x^3 \ln x}$

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5. Consider the initial value problem

$$\begin{cases} y' = -2y + t; \\ y(0) = 1. \end{cases}$$

(a.) Write down the Picard operator $\mathcal{P}f$ corresponding to this IVP.

$$\mathcal{P}f(\varphi) = \int_0^t -2\varphi + s \, ds + 1$$

(b.) Let $\varphi_0(t) = 1$. Use Picard's iterative method to calculate φ_1 and φ_2.

$$\varphi_0 = 1$$

$$\varphi_1 = \mathcal{P}f(\varphi_0) = \int_0^t -2 + s \, ds + 1$$

$$= -2t + \frac{1}{2}t^2 + 1$$

$$\boxed{\varphi_1 = \frac{1}{2}t^2 - 2t + 1}$$

$$\varphi_2 = \mathcal{P}f(\varphi_1) = \int_0^t -2\left(\frac{1}{2}s^2 - 2s + 1\right) + s \, ds + 1$$

$$= \int_0^t -s^2 + 4s + s - 2 \, ds + 1$$

$$= \int_0^t -s^2 + 5s - 2 \, ds + 1$$

$$\boxed{\varphi_2 = -\frac{1}{3}t^3 + \frac{5}{2}t^2 - 2t + 1}$$

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6. Solve the initial value problem

$$\begin{cases} y' = \dfrac{\sqrt{1-y^2}}{1+x^2}; \\ y(0) = 0. \end{cases}$$

Solve for y explicitly.

Hint: $\frac{d}{dt}[\sin^{-1}(t)] = \frac{1}{\sqrt{1-t^2}}$ and $\frac{d}{dt}[\tan^{-1}(t)] = \frac{1}{1+t^2}$

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2} \Rightarrow \frac{1}{\sqrt{1-y^2}}\,dy = \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \sin^{-1}(y) = \tan^{-1}(x) + C$$

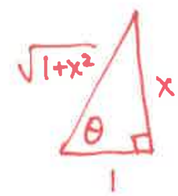
Plugging in $y(0)=0$ $\Rightarrow$ $0 = 0 + C$ $\Rightarrow$ $C = 0$.

So $y = \sin(\tan^{-1}(x))$

Put $\theta = \tan^{-1}(x)$

$\Rightarrow \tan\theta = \frac{x}{1}$

then $\sin\theta = \frac{x}{\sqrt{1+x^2}}$



$$\boxed{y(x) = \frac{x}{\sqrt{1+x^2}}}$$

Bonus. [5 points] What is the most interesting thing that you learned this summer?

A full grown platypus weighs only 2 lbs. It is a mammal, but it lays eggs. Its back feet have poisonous barbs. And its tail is not used for swimming, but to store fat.

WHOA!