Math 555: Differential Equations
Final Exam, Part I

Thursday, 24 July 2014

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 3 problems in the space provided. Show enough work.

1. Use the <u>definition</u> of Laplace transform to calculate

$$F(s) = \mathcal{L}\{t^2 e^t\}.$$

Be sure to treat all improper integrals properly, and determine the domain of F.

$$F(s) = \mathcal{L}\{t^2 e^t\} = \int_0^\infty e^{-st} t^2 e^t \, dt$$

$$= \int_0^\infty t^2 e^{-(s-1)t} \, dt$$

$$= \lim_{A \to \infty} \left[-\frac{t^2}{s-1} e^{-(s-1)t} - \frac{2t}{(s-1)^2} e^{-(s-1)t} - \frac{2}{(s-1)^3} e^{-(s-1)t} \right]_0^A - 0$$

u	dv
$+\ t^2$	e^{-ut}
$-\ 2t$	$-\frac{1}{u} e^{-ut}$
$+\ 2$	$\frac{1}{u^2} e^{-ut}$
	$-\frac{1}{u^3} e^{-ut}$

this $u = (s-1)$

$$= \lim_{A \to \infty} \left[\frac{-A^2}{s-1} e^{-(s-1)A} - \frac{2A}{(s-1)^2} e^{-(s-1)A} - \frac{2}{(s-1)^3} e^{-(s-1)A} \right] + \frac{2}{(s-1)^3}$$

Each of these limits converges to 0 so long as $s-1 > 0$, or $s > 1$.

Thus $\boxed{F(s) = \frac{2}{(s-1)^3}, \quad s > 1}$

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2. Use Laplace transforms to solve the initial value problem,

$$\begin{cases} 2y'' + 4y' - 6y = 0; \\ y(0) = 1, \\ y'(0) = -1. \end{cases}$$

$$2\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

$$2(s^2 Y - sy(0) - y'(0)) + 4(sY - y(0)) - 6Y = 0$$

$$(2s^2 + 4s - 6)Y \quad -2s + 2 - 4 = 0$$

$$Y(s) = \frac{2s+2}{2(s^2+2s-3)} = \frac{s+1}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

So $\quad y(t) = Ae^{-3t} + Be^{+t}$

$$s+1 = A(s-1) + B(s+3)$$

$s=1$: $\quad 2 = 4B \Rightarrow B = \frac{1}{2}$

$s=-3$: $\quad -2 = -4A \Rightarrow A = \frac{1}{2}$

so $\quad \boxed{y(t) = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{t}}$

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3. Use Laplace transforms to solve the initial value problem,

$$\begin{cases} y'' - 2y' + y = u_{\frac{\pi}{2}}(t)\sin(t); \\ y(0) = y'(0) = 0. \end{cases}$$

[Hint: $\sin(t) = \cos(\frac{\pi}{2} - t)$.] $= \cos(t - \pi/2)$ because cosine is <u>even</u>.

$$y'' - 2y' + y = u_{\frac{\pi}{2}}(t)\cos(t - \tfrac{\pi}{2})$$

Apply $\mathcal{L}$:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$$

$$s^2Y - 2Y + Y = \frac{e^{-\frac{\pi}{2}s} \cdot s}{s^2+1}$$

so $Y(s) = e^{-\pi/2 s} \cdot \dfrac{s}{(s^2+1)(s-1)^2} = e^{-\pi/2 s} H(s)$

then $y(t) = u_{\frac{\pi}{2}}(t)\, h(t - \pi/2)$ where $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

$$H(s) = \frac{s}{(s^2+1)(s-1)^2} = \frac{As+B}{(s^2+1)} + \frac{C}{s-1} + \frac{D}{(s-1)^2} = A\frac{s}{s^2+1} + B\frac{1}{s^2+1} + C\frac{1}{s-1} + D\frac{1}{(s-1)^2}$$

Thus $h(t) = A\cos t + B\sin t + Ce^t + Dte^t$.

$$s = (As+B)(s-1)^2 + C(s^2+1)(s-1) + D(s^2+1) = (As+B)(s^2-2s+1) + C(s^3-s^2+s-1) + \tfrac{1}{2}s^2 + \tfrac{1}{2}$$

$s=1$: $1 = 2D \Rightarrow \boxed{D = \frac{1}{2}}$

otherwise: $s = As^3 - 2As^2 + As + Bs^2 - 2Bs + B + Cs^3 - Cs^2 + Cs - C + \frac{1}{2}s^2 + \frac{1}{2}$

$A + C = 0 \Rightarrow A = -C \Rightarrow C = 0$

$-2A + B - C = -1/2 \Rightarrow -A + B = -1/2 \Rightarrow A = 0$

$A - 2B + C = 1 \qquad -2B = 1 \Rightarrow B = -\frac{1}{2}$

$B - C = -1/2$

So finally...

$$\boxed{\begin{array}{l} y(t) = u_{\pi/2}(t)\, h(t - \pi/2) \text{ where} \\ h(t) = -\frac{1}{2}\sin t - \frac{1}{2}te^t \end{array}}$$

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Part II. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

4. Prove the theorem.

Theorem. *If* $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > a \geq 0$*, and if* $c \in \mathbb{R}$*, then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c.$$

Proof. $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt$

$= \int_0^\infty e^{-(s-c)t}f(t)\,dt \quad \{\text{put } u = s-c\}$

$= \int_0^\infty e^{-ut}f(t)\,dt$

$= F(u)$

$= F(s-c).$ $\square$

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5. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, where

$$F(s) = \frac{e^{-s} - se^{-2s}}{s^3 - s^2 + s - 1}$$

$$F(s) = e^{-s}\frac{1}{(s^2+1)(s-1)} - e^{-2s}\frac{s}{(s^2+1)(s-1)}$$

$$\frac{\xi}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$\xi = (As+B)(s-1) + C(s^2+1)$$

$$\xi = (A+C)s^2 + (B-A)s + (C-B)$$

$\xi = 1$:

$A+C=0$

$B-A=0 \Rightarrow B=A$

$C-B=1 \Rightarrow C-A=1$

$\Rightarrow 2C=1$

so $C=\frac{1}{2}$

$B=\frac{-1}{2}$

$A=\frac{-1}{2}$

$\xi = s$:

$A+C=0 \Rightarrow A+B=0$

$B-A=1$

$C-B=0 \Rightarrow C=B$

$2B=1 \Rightarrow B=\frac{1}{2}$

$C=\frac{1}{2}$

$A=\frac{-1}{2}$

$$\text{so } F(s) = e^{-s}\left(\frac{-1}{2}\frac{s+1}{s^2+1} + \frac{1}{2}\frac{1}{s-1}\right) - e^{-2s}\left(\frac{-1}{2}\frac{s-1}{s^2+1} + \frac{1}{2}\frac{1}{s-1}\right)$$

$$\Rightarrow \boxed{f(t) = u_1(t)\left[\frac{-1}{2}\cos(t-1) - \frac{1}{2}\sin(t-1) + \frac{1}{2}e^{t-1}\right] + u_2(t)\left[\frac{1}{2}\cos(t-2) - \frac{1}{2}\sin(t-2) + \frac{1}{2}e^{t-2}\right]}$$

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6. Write the piecewise function g as a linear combination of step functions, then find its Laplace transform.

$$g(t) = \begin{cases} 0 & 0 \le t < 3 \\ 3 & 3 \le t < 5 \\ -1 & 5 \le t < 6 \\ 1 & 6 \le t < 10 \\ 0 & 10 \le t \end{cases}$$

$g(t) = 3u_3(t) - 4u_5(t) + 2u_6(t) - u_{10}(t)$

$G(s) = \mathcal{L}\{g\} = \dfrac{3e^{-3s} - 4e^{-5s} + 2e^{-6s} - e^{-10s}}{s}$