Math 555: Differential Equations
Midterm Exam 3

Friday, 11 July 2014

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 3 problems in the space provided. Show enough work.

1. Consider the second-order differential equation

$$y'' - 3xy' + y = 0.$$

Find a power series solution centered at $x_0 = 0$.

(*a*.) [12] Find and clearly identify the recurrence relation.

(*b*.) [6] Find the first four terms of each fundamental solution. Clearly label each solution.

(*c*.) [2] Write down and clearly label the general solution. (This does <u>not</u> have to be in Σ-notation.)

You may use the back of the page for work as well, if needed.

$$y'' - 3xy' + y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 3n\, a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$(2a_2 + a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 3n\, a_n + a_n \right] x^n = 0$$

a.) $\boxed{a_2 = -\frac{1}{2} a_0}$ and $\boxed{a_{n+2} = \frac{(3n-1)\, a_n}{(n+2)(n+1)}}$ $n \geq 1$

a_0

$a_2 = -\frac{1}{2} a_0$

$a_4 = \frac{5 a_2}{4 \cdot 3} = \frac{-5}{4!} a_0$

$a_6 = \frac{11 a_4}{6 \cdot 5} = \frac{-11 \cdot 5}{6!} a_0$

a_1

$a_3 = \frac{2 a_1}{3!}$

$a_5 = \frac{8 \cdot a_3}{5 \cdot 4} = \frac{8 \cdot 2\, a_1}{5!}$

$a_7 = \frac{14 \cdot a_5}{7 \cdot 6} = \frac{14 \cdot 8 \cdot 2\, a_1}{7!}$

b) $\boxed{y_1 = a_0 - \frac{1}{2} a_0 x^2 - \frac{5}{4!} a_0 x^4 - \frac{11 \cdot 5}{6!} x^6 + \cdots}$

$\boxed{y_2 = a_1 x + \frac{2}{3!} a_1 x^3 + \frac{8 \cdot 2}{5!} a_1 x^5 + \frac{14 \cdot 8 \cdot 2}{7!} a_1 x^7 + \cdots}$

So,

c.) $\boxed{y = a_0\left(1 - \frac{1}{2}x^2 - \frac{5}{4!}x^4 - \frac{11 \cdot 5}{6!}x^6 + \cdots\right) + a_1\left(x + \frac{2}{3!}x^3 + \frac{8 \cdot 2}{5!}x^5 + \frac{14 \cdot 8 \cdot 2}{7!}x^7 + \cdots\right)}$

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2. (*a.*) [15] Use your favorite method to find the power series expansion of $f(x) = \ln(x+2)$ about the point $x_0 = -1$.

$$f(x) = \ln(x+2)$$

$$f'(x) = \frac{1}{x+2} = \frac{1}{(x+1)+1} = \frac{1}{1-(-(x+1))} = \sum_{n=0}^{\infty} (-(x+1))^n = \sum_{n=0}^{\infty} (-1)^n (x+1)^n$$

$$f(x) = \int f'(x)\,dx = C + \sum_{n=0}^{\infty} \int (-1)^n (x+1)^n\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x+1)^{n+1}$$

$f(-1) = C$ and $f(-1) = \ln(-1+2) = \ln(1) = 0$, so $C = 0$

Thus

$$\boxed{\ln(x+2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x+1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x+1)^n}$$

(*b.*) [5] What are the radius and interval of convergence? Justify your answer.

$R = 1$ b/c it came from a geometric series!

$I: (-2, 0)$ again from geometric.

otherwise, you need to use some "tests".

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3. Determine the general term a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 5\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. (*a.*) [18] Write the series $\sum_{n=0}^{\infty} a_n x^n$ that solves this equation. (*b.*) [2] What function is represented by your solution?

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 5\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} 5 a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 5 a_n \right] x^n = 0.$$

so $\boxed{a_{n+1} = \frac{5}{n+1} a_n}$ R.R.

$a_0 = ?$

$a_1 = 5 a_0$

$a_2 = \frac{5}{2} a_1 = \frac{5^2}{2 \cdot 1} a_0$

$a_3 = \frac{5}{3} a_2 = \frac{5^3}{3!} a_0$

$\vdots$

$\boxed{a_n = \frac{5^n}{n!} a_0}$

The soln is $\sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n = \boxed{a_0 \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}}$ a.)

b.) $\boxed{a_0 \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = a_0 e^{5x}}$

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Part II. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

4. Find the general solution of the equation

$$(x-1)^2 y'' + 5(x-1)y' + 3y = 0$$

that is valid in any interval not containing the point $x_0 = 1$.

let $u = x-1$:

$$u^2 y'' + 5u y' + 3y = 0$$

Euler! Char. eq'n:

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$r = -1, -3$$

So $y(u) = C_1 u^{-1} + C_2 u^{-3}$

and

$\boxed{y(x) = C_1 (x-1)^{-1} + C_2 (x-1)^{-3}}$ or $y(x) = \dfrac{C_1}{x-1} + \dfrac{C_2}{(x-1)^3}$

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5. Consider the second-order differential equation

$$(x^3+x^2+x)y''+2xy'-5y=0.$$

Determine the minimum radius of convergence of the power series solution centered about the points:

(*a.*) [6.5] $x_0=-\frac{1}{2}$

$x^3+x^2+x=0$

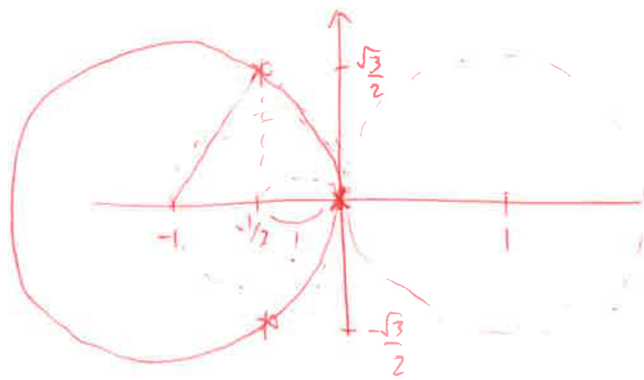
$x(x^2+x+1)=0$

$x=0 \quad (x+\frac{1}{2})^2=-\frac{3}{4}$

$x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$R=\frac{1}{2}$

(*b.*) [6.5] $x_0=-1$



$R=1$

(*c.*) [6.5] $x_0=1$

$R=1$

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6. Consider the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad \alpha \in \mathbb{R}.$$

(*a.*) [10] Find all singular points and determine whether each one is regular or irregular.

$\boxed{x_0 = \pm 1}$

$x_0 = 1$: $\lim_{x\to 1} (x-1)\frac{-2x}{1-x^2} = \lim_{x\to 1} \cancel{(x-1)}\frac{-2x}{\cancel{(1-x)}(1+x)} = \frac{2}{2} = 1$

$\lim_{x\to 1} (x-1)^2 \frac{\alpha(\alpha+1)}{1-x^2} = \lim_{x\to 1} \frac{(x-1)^2 \cdot -\alpha(\alpha+1)}{\cancel{(1-x)}(1+x)} = \frac{0}{2} = 0.$

So $x_0 = 1$ is regular.

$x_0 = -1$: $\lim_{x\to -1} \cancel{(x+1)} \frac{-2x}{(1-x)\cancel{(1+x)}} = \frac{2}{2} = 1$

$\lim_{x\to -1} (x+1)^2 \frac{\alpha(\alpha+1)}{(1-x)\cancel{(1+x)}} = \frac{0}{2} = 0$

so $x_0 = -1$ is also regular.

(*b.*) [10] Determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point.

$P(x) = 1 - x^2 \qquad P(1/\xi) = 1 - 1/\xi^2$

$Q(x) = -2x \qquad Q(1/\xi) = \frac{-2}{\xi}$

$R(x) = \alpha(\alpha+1) \qquad R(1/\xi) = \alpha(\alpha+1)$

$p(\xi) = \frac{1}{\xi^2(1-1/\xi^2)}\left(2\xi(1-1/\xi^2) + \frac{2}{\xi}\right) = \frac{1}{\xi^2-1}(2\xi) = \frac{2\xi}{\xi^2-1}$ ok at $\xi = 0$.

$q(\xi) = \frac{\alpha(\alpha+1)}{\xi^4(1-1/\xi^2)} = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)}$ not ok at $\xi = 0$, so x_∞ is singular.

$\xi p(\xi)$ is ok since $p(\xi)$ is.

$\xi^2 q(\xi) = \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)} = \frac{\alpha(\alpha+1)}{\xi^2-1}$ is ok at $\xi = 0$, so

$\boxed{x_\infty \text{ is regular.}}$