Math 555: Differential Equations
Midterm Exam 2

Friday, 27 June 2014

Name: Key

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 3 problems in the space provided. Show enough work.

1. Solve the initial value problem

$$\begin{cases} y'' + 4y' + 3y = 0, \\ y(0) = 2, \\ y'(0) = -1. \end{cases}$$

ok

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r = -1, -3$

$y = C_1 e^{-t} + C_2 e^{-3t}$

$y' = -C_1 e^{-t} - 3C_2 e^{-3t}$

$y(0) = C_1 + C_2 = 2$

$y'(0) = -C_1 - 3C_2 = -1$

$-2c_2 = 1 \Rightarrow C_2 = \frac{-1}{2}$

$C_1 = 2 + \frac{1}{2} = \frac{5}{2}$

Thus, $\boxed{y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}}$

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2. Consider the equation $t^2y'' + ty' - 4y = 0$, for $t > 0$. Supposing $y_1(t) = t^2$ is a solution, use the method of reduction of order to find a second solution, $y_2(t)$. Clearly identify $y_2(t)$.

put $y = t^2 v$ where $v = v(t)$

$y' = t^2 v' + 2tv$

$y'' = t^2 v'' + 4tv' + 2v$

ok

Plugging in:

$t^2y'' + ty' - 4y = t^2(t^2v'' + 4tv' + 2v) + t(t^2v' + 2tv) - 4t^2v = 0$

$\Rightarrow t^4v'' + 4t^3v' + \cancel{2t^2v} + t^3v' + \cancel{2t^2v} - \cancel{4t^2v} = 0$

$\Rightarrow t^4v'' + 5t^3v' = 0$

$\Rightarrow v'' + \frac{5}{t}v' = 0 \Rightarrow (v')' + \frac{5}{t}v' = 0$

$\mu(t) = e^{5\int \frac{1}{t}\,dt} = t^5$ and $v' = \frac{C_2}{\mu(t)}$

so $v' = C_2 t^{-5} \Rightarrow v = C_2 t^{-4} + C_1$

and $y = t^2(C_2 t^{-4} + C_1) = C_2 t^{-2} + C_1 t^2$

and $\boxed{y_2(t) = t^{-2}}$

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Find the general sol'n.

3. ~~Solve the initial value problem~~

$$\begin{cases} y'' - 2y' + 5y = 8\sin x - 4\cos x, \\ \cancel{y(0) = 3,} \\ \cancel{y'(0) = 9.} \end{cases}$$

homog.: $y'' - 2y' + 5y = 0$

$r^2 - 2r + 5 = 0$

$(r-1)^2 = -4$

$r = 1 \pm 2i$

$y_h = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$

Guess: $Y(t) = A\sin t + B\cos t$

$Y'(t) = A\cos t - B\sin t$

$Y''(t) = -A\sin t - B\cos t$

Plug in: $Y'' - 2Y' + 5Y = -A\sin t - B\cos t - 2A\cos t + 2B\sin t + 5A\sin t + 5B\cos t$

$= (4A + 2B)\sin t + (-2A + 4B)\cos t = 8\sin x - 4\cos x$

so $4A + 2B = 8$

$-2A + 4B = -4$

$10B = 0$

$\Rightarrow B = 0$

$\Rightarrow A = 2.$

so $\boxed{y = 2\sin t + C_1 e^t \cos(2t) + C_2 e^t \sin(2t)}$

$\cancel{y' = 2\cos t + C_1 e^t \cos(2t) - 2C_1 e^t \sin(2t) + C_2 e^t \sin(2t) + 2C_2 e^t \cos(2t)}$

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Part II. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

4. (*a.*) Find the general solution y_h of the homogeneous equation

$$y'' - 4y' + 4y = 0.$$

$r^2 - 4r + 4 = 0$

ok

$r = 2, 2$

$$\boxed{y_h = C_1 e^{2t} + C_2 t e^{2t}}$$

(*b.*) Determine a suitable form for the particular solution of the non-homogeneous DE, if you were to use the method of undetermined coefficients. Do not solve for the coefficients.

$$y'' - 4y' + 4y = t^2 e^{2t} + 4te^t \sin(t).$$

$Y(t) = t^2(At^2 + Bt + C)e^{2t} + (Dt + E)e^t \sin t + (Ft + G)e^t \cos(t).$

$\uparrow$

b/c homog. !

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5. Find the general solution of the differential equation,

$$y'' + y = \tan(t), \quad 0 < t < \pi/2.$$

homog. $y'' + y = 0$, $r^2 + 1 = 0$, $r = \pm i$

$y_h = C_1 \cos t + C_2 \sin t$

ok

$y_1 = \cos t$

$y_2 = \sin t$

$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

$g(t) = \tan(t).$

V.o.P.: $y(t) = -\cos t \underbrace{\int_{t_0}^{t} \sin(s)\tan(s)\,ds}_{I_1} + \sin t \underbrace{\int_{t_0}^{t} \cos(s)\tan(s)\,ds}_{I_2},$

$I_1 = \int_{t_0}^{t} \frac{\sin^2(s)}{\cos(s)}\,ds = \int_{t_0}^{t} \frac{1-\cos^2(s)}{\cos(s)}\,ds = \int_{t_0}^{t} \sec(s)\,ds - \int_{t_0}^{t} \cos(s)\,ds = \ln|\sec t + \tan t| - \sin t + C_1$

$I_2 = \int_{t_0}^{t} \sin(s)\,ds = -\cos(s)\Big|_{t_0}^{t} = -\cos t + C_2$

$y(t) = -\cos t\left[\ln|\sec t + \tan t| - \sin t + C_1\right] + \sin t(-\cos t) + C_2 \sin t$

$= -\cos t \cdot \ln|\sec t + \tan t| + \cancel{\cos t \sin t} - \cancel{\cos t \sin t} + C_1 \cos t + C_2 \sin t$

so $\boxed{y(t) = -\cos t \cdot \ln|\sec t + \tan t| + C_1 \cos t + C_2 \sin t}$

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6. Solve the initial value problem

$$\begin{cases} t^2 y'' - 7ty' + 7y = 0, \\ y(1) = 1, \\ y'(1) = -1. \end{cases}$$

!

Euler $\longrightarrow$ $y'' - 8y' + 7y = 0$

$r^2 - 8r + 7 = 0$

$(r^2 - 8r + 16) - 9 = 0$

$(r-4)^2 = 9$

$r = 4 \pm 3$

$r = 1, 7$

thus $y(x) = C_1 e^{x} + C_2 e^{7x}$

$y(t) = C_1 t + C_2 t^7$

$y' = C_1 + 7C_2 t^6$

$y(1) = C_1 + C_2 = 1$

$y'(1) = C_1 + 7C_2 = -1$

$-6C_2 = 2$

$C_2 = -\frac{1}{3}$

$C_1 = \frac{4}{3}$

Thus, the particular sol'n is:

$$\boxed{y(t) = \frac{4}{3}t - \frac{1}{3}t^7}$$