Math 555: Differential Equations
Midterm Exam 1

Friday, 13 June 2014

Name: KEY

Instructions: Complete all 3 problems in part I, and 2 of the 3 problems in part II. Clearly mark the problem in part II that you would like to omit. Each completed problem is worth 20 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

Name:________________________________

Part I. Complete all 3 problems in the space provided. Show enough work.

1. Find the general solution of the differential equation.

$$(2y + x^2 y)y' = x$$

$$y(2+x^2)\frac{dy}{dx} = x$$

$$\rightarrow \quad y\,dy = \frac{x}{2+x^2}\,dx$$

so this eq'n is separable.

$$\int y\,dy = \int \frac{x}{2+x^2}\,dx \qquad u = (2+x^2), \quad du = 2x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u) + C$$

so $\boxed{y^2 = \ln(2+x^2) + C}$

Name:______________________________

2. Solve the initial value problem.

$$\begin{cases} y' = 2ty^2 \\ y(0) = -1. \end{cases}$$

Write your answer y as a function of t. Be sure to include the domain of the solution.

<u>Separable:</u>

$$\frac{dy}{y^2} = 2t\,dt$$

$$\int_{-1}^{y} \frac{1}{s^2}\,ds = \int_0^t 2s\,ds$$

$$-\frac{1}{s}\Big|_{-1}^{y} = s^2\Big|_0^t$$

$$-\frac{1}{y} - 1 = t^2$$

$$-\frac{1}{y} = t^2 + 1$$

$$\boxed{y = \frac{-1}{1+t^2}, \quad}$$

since $t_0 = 0$, domain of the solution is $t \in \cancel{(-1,1)}\ \mathbb{R}$.

Name:______________________________

3. Consider the differential equation

$$(2y+1) + \left(x - \frac{y}{x}\right) y' = 0.$$

(*a*.) [5 points] Show that this equation is *not* exact.

$$\left.\begin{aligned} \partial_y M &= \partial_y (2y+1) = 2. \\ \partial_x N &= \partial_x \left(x - \frac{y}{x}\right) = 1 + \frac{y}{x^2} \end{aligned}\right\} \neq$$

$\partial_y M \neq \partial_x N$, so the DE is not exact.

(*b*.) [15 points] Use the integrating factor $\mu(x, y) = x$ to find the general solution of the differential equation. (Don't forget to verify that the new equation is exact.)

$$\to \quad (2xy + x)\, dx + (x^2 - y)\, dy = 0$$

$$\left.\begin{aligned} \partial_y M &= 2x \\ \partial_x N &= 2x \end{aligned}\right\} = \quad \text{so the new DE is exact.}$$

$$\psi(x,y) = \int 2xy + x \, dx = x^2 y + \frac{1}{2}x^2 + f(y)$$

$$\psi(x,y) = \int x^2 - y \, dy = x^2 y - \frac{1}{2}y^2 + h(x)$$

Putting it together:

$$\psi(x,y) = \boxed{x^2 y + \frac{1}{2}x^2 - \frac{1}{2}y^2 = C}$$

or

$$x^2 + 2x^2 y - y^2 = C$$

Name:________________________________

Part II. Complete 2 of the 3 problems. Show enough work. Clearly mark the one problem that you wish to omit.

4. (*a.*) [10 points] Find the domain of the solution of the initial value problem *without* solving the equation explicitly.

$$\begin{cases} \ln(t)y' + \overset{1}{\cancel{\sin(t)}}y = \tan(t), \\ y(\overset{\pi/3}{\cancel{\pi}}) = -1. \end{cases} \longrightarrow \quad y' + \frac{\sin(t)}{\ln(t)}y = \frac{\tan(t)}{\ln(t)}$$

domain of $\ln(t)$: $(0,\infty)$ and $\ln(t) = 0$ if $t = 1$.

domain of $\tan(t)$: $\cdots\cup(-\frac{\pi}{2},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2})\cup\cdots$

domain of $\sin(t)$: $(-\infty,\infty)$

so $\text{dom}(p) = (0,1)\cup(1,\infty)$ $\swarrow \pi/3$

$\text{dom}(g) = [(0,1)\cup(1,\infty)] \cap [\cdots\cup(-\pi/2,\pi/2)\cup(\pi/2,3\pi/2)\cup\cdots]$ $\pi/3$

so domain of the solution is $\boxed{(1,\pi/2)}$

(*b.*) [10 points] Does the initial value problem have a solution? Justify your answer.

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t^2 - 2ty + y^2}{\tan(t)}, \\ y(\pi/4) = 0. \end{cases}$$

$f(t,y) = \dfrac{t^2 - 2ty + y^2}{\tan(t)}$ $\qquad$ $\text{dom}(f)$: $t \neq \pm\frac{\pi}{2} + n\pi$

so f is continuous at $\pi/4 = t$.

$\partial_y f(t,y) = \dfrac{2y - 2t}{\tan t}$ has the same domain, so it's also continuous at $t = \pi/4$.

Thus there exists a unique sol'n passing through the pt. $(\pi/4, 0)$.

Name:______________________________

5. Consider the initial value problem

$$\begin{cases} y' = y(t-1), \\ y(0) = \cancel{0}\,1 \end{cases}$$

(a.) [5 points] Write the Picard operator $\mathcal{P}f(y)$ corresponding to this IVP.

$$\mathcal{P}f(y) = \int_0^t y(s-1)\,ds + 1$$

(b.) [15 points] Letting $\varphi_0(t) = \cancel{0}\,1$, use Picard's iterative method to calculate $\varphi_1(t)$, and $\varphi_2(t)$~~, and $\varphi_3(t)$~~. Clearly label each one.

$$\varphi_0 = 1$$

$$\varphi_1 = \mathcal{P}f(\varphi_0) = \int_0^t (s-1)\,ds + 1 = \frac{1}{2}s^2 - s\Big|_0^t + 1 = \boxed{\frac{1}{2}t^2 - t + 1 = \varphi_1}$$

$$\varphi_2 = \mathcal{P}f(\varphi_1) = \int_0^t \left(\frac{1}{2}s^2 - s + 1\right)(s-1)\,ds + 1$$

$$= \int_0^t \frac{1}{2}s^3 - s^2 + s - \frac{1}{2}s^2 + s - 1\,ds + 1$$

$$= \int_0^t \frac{1}{2}s^3 - \frac{3}{2}s^2 + 2s - 1\,ds + 1$$

$$= \left(\frac{1}{8}s^4 - \frac{1}{2}s^3 + s^2 - s\right)\Big|_0^t + 1$$

$$= \boxed{\frac{1}{8}t^4 - \frac{1}{2}t^3 + t^2 - t + 1 = \varphi_2}$$

Name:______________________________

6. Solve the initial value problem

$$\begin{cases} y' + \frac{3}{t} y = 2e^{t^2}, \\ y(1) = 2 \end{cases}$$

~~makes no sense~~ !

$p(t) = \frac{3}{t}$

$g(t) = 2e^{t^2}$

$\mu(t) = e^{\int \frac{3}{t} dt} = e^{3\ln t} = t^3$

$$y(t) = \frac{1}{t^3} \underbrace{\int_1^t 2s^3 e^{s^2} ds} + \frac{2}{t^3}$$

$\int_1^t 2s^3 e^{s^2} ds \qquad u = s^2 \quad du = 2s\,ds \qquad u(t) = t^2 \quad u(1) = 1$

$$= \int_1^{t^2} ue^u du = (ue^u - e^u)\Big|_1^{t^2} = t^2e^{t^2} - e^{t^2} - \cancel{(e - e)} = t^2e^{t^2} - e^{t^2} \cancel{+1} + 0$$

so

$$y(t) = \frac{1}{t^3}\left[t^2e^{t^2} - e^{t^2}\right] + \frac{2}{t^3} = \frac{e^{t^2}}{t} - \frac{e^{t^2}}{t^3} + \frac{2}{t^3}$$

or

$$\boxed{y(t) = \frac{2 + t^2e^{t^2} - e^{t^2}}{t^3}, \quad t > 0}$$