3.4. Basis and Dimension

Defn. The vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ form a basis for a vector space V if and only if

i.) $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent

ii.) $\underline{v}_1, \dots, \underline{v}_n$ span V.

Ex. The standard basis for $\mathbb{R}^3$ is $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$; however, there are many other choices of bases of $\mathbb{R}^3$.

e.g.,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Ex. The matrices $E_{11}, E_{12}, E_{21}, E_{22}$ form a basis for $\mathbb{R}^{2\times 2}$.

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Thm. If $\{\underline{v}_1, \dots, \underline{v}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, $m > n$, is linearly dependent.

Proof Let $\underline{u}_1, \dots, \underline{u}_m$ be m vectors in V. Since $\underline{v}_1, \dots, \underline{v}_n$ span V, then

$$\underline{u}_i = a_{i1}\underline{v}_1 + a_{i2}\underline{v}_2 + \dots + a_{in}\underline{v}_n \qquad i = 1, 2, \dots, m$$

Any linear combination $c_1\underline{u}_1 + c_2\underline{u}_2 + \dots + c_m\underline{u}_m$ can be written

$$c_1 \sum_{j=1}^{n} a_{1j}\underline{v}_j + c_2 \sum_{j=1}^{n} a_{2j}\underline{v}_j + \dots + c_m \sum_{j=1}^{n} a_{mj}\underline{v}_j$$

Rearranging the terms we obtain

$$c_1\underline{u}_1 + \dots + c_m\underline{u}_m = \sum_{i=1}^{m}\left[c_i \sum_{j=1}^{n} a_{ij}\underline{v}_j\right] = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}c_i\right)\underline{v}_j$$

Now consider the system of equations

$$\sum_{i=1}^{m} a_{ij}c_i = 0 \qquad j = 1, 2, \dots, n$$

This represents a homogeneous system w/ more unknowns than equations. Thus it must have a nontrivial solution (why?). Therefore $\underline{u}_1, \dots, \underline{u}_m$ are linearly dependent. □

<u>Corollary.</u> If $\{\underline{v}_1, \dots, \underline{v}_n\}$ and $\{\underline{u}_1, \dots, \underline{u}_m\}$ are both bases for V, then $m = n$. □

<u>Def'n.</u> Let V be a vector space. If V has a basis consisting of n vectors, we say that V has <u>dimension</u> n.

The subspace $\{0\}$ has dimension 0.

V is said to be <u>finite-dimensional</u> if there is a finite set of vectors that spans V. Otherwise V is <u>infinite-dimensional</u>.

<u>Ex.</u> Show that $\{1, x, x^2, \dots, x^{n-1}\}$ form a basis for P_n.

<u>Ex.</u> Let P be the space of all polynomials. Deduce that P is infinite dimensional by showing that $W[1, x, x^2, \dots, x^n] > 0$ for all $n \in \mathbb{N}$. The same argument shows that $C[a,b]$ is infinite dimensional.

Thm. If V is a vector space of dimension $n>0$, then
i.) any set of n linearly independent spans V
ii.) any n vectors that span V are linearly independent

Proof. Let $\underline{v}_1, \dots, \underline{v}_n$ be linearly independent. Since $\dim V = n$, it follows that $\underline{v}_1, \dots, \underline{v}_n, \underline{v}$ must be linearly dependent. Thus, there exist $c_1, \dots, c_n, c_{n+1}$ not all 0 s.t.

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n + c_{n+1} \underline{v} = \underline{0}.$$

Now $c_{n+1} \neq 0$ because otherwise $\underline{v}_1, \dots, \underline{v}_n$ would be linearly dependent. Without loss of generality, put $c_{n+1} = 1$. (Prove that it's ok to do this.)
Then, solving for $\underline{v}$, we obtain

$$\underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n.$$

This proves i.)

Now suppose that $\underline{v}_1, \dots, \underline{v}_n$ span V. If $\underline{v}_1, \dots, \underline{v}_n$ are linearly dependent the one, say $\underline{v}_n$, can be written as a linear combination of the others. Then $\underline{v}_1, \dots, \underline{v}_{n-1}$ will still span V. But this contradicts $\dim V = n$. $\square$

Ex. Show that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Thm. Let V be a vector space w/ $\dim V = n > 0$.
i) No set of fewer than n vectors will span V.
ii) any set of fewer than n lin. ind. vectors can be extended to a basis.
iii) any spanning set containing more than n vectors can be pared down to form a basis for V.

In other words, a basis is a minimal spanning set of V, and a maximal linearly independent set of vectors of V.

Proof. p. 142 □

RE.14. Find the dimension of the subspace of P_3 spanned by $x, x-1, x^2+1$

Apply the isomorphism Γ: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = 1$$

So these vectors are linearly independent, hence span all of $\mathbb{R}^3$. Reapplying Γ (really, Γ^{-1}) this means that $\{x, x-1, x^2+1\}$ span all of P_3.

RE.15. Let S be the subspace of P_3 satisfying $p(0)=0$ and T be the subspace of all polynomials satisfying $q(1)=0$.

Find bases for S, T, and $S \cap T$.

S: $p = ax^2 + bx + c$, $p(0) = c \Rightarrow c = 0$.
thus $p = ax^2 + bx$ and $\{x^2, x\}$ form a basis for S,

T: $q = a_1 + a_2(x-1) + a_3(x-1)^2 + \cdots + a_n(x-1)^{n-1} + \cdots$
$q(1) = 0 \Rightarrow a_1 = 0$
so a basis is $\{(x-1), (x-1)^2, (x-1)^3, \ldots, (x-1)^n, \ldots\}$

$S \cap T$: $p(1) = a + b = 0 \Rightarrow a = -b \Rightarrow p = ax^2 - ax = a\boxed{(x^2 - x)}$ basis for $S \cap T$.