3.2. Subspaces

Let V be a vector space, and let S be a subset of V. If S is itself a vector space, then S is called a (vector) subspace of V.

Since all of the elements of V satisfy the eight "A axioms", we need only verify that the elements of S verify the two closure axioms.

Def'n. Let S be a nonempty subset of a vector space V. If S satisfies

C1. $\alpha \underline{x} \in S$ whenever $\underline{x} \in S$ and $\alpha \in \mathbb{R}$, and

C2. $\underline{x} + \underline{y} \in S$ whenever $\underline{x}, \underline{y} \in S$,

then S is said to be a (vector) subspace of V.

Remarks.

1. In a vector space V, the sets $\{\underline{0}\}$ and V are subspaces, called trivial subspaces. All other subspaces are called proper subspaces.

2. For S to be a subspace, it must be nonempty. To verify that S is nonempty, it is usually easiest to show that $\underline{0} \in S$.
(${\underline{0}}$ must be an element of every subspace of V.)

Ex. $S = \{(x_1, x_2, x_3)^T \mid x_1 = x_2\}$

S is nonempty since $(0,0,0)^T \in S$.

C1. $(a,a,b) \in S$, $\alpha \in \mathbb{R}$, $\alpha(a,a,b) = (\alpha a, \alpha a, \alpha b) \in S$.

C2. $(a,a,b) + (c,c,d) = (a+c, a+c, b+d) \in S$.

Therefore S is a subspace of $\mathbb{R}^3$.

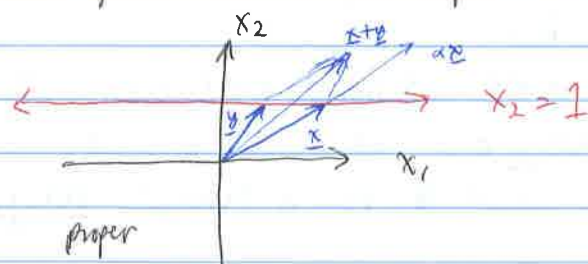
Ex. $S = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$

$$\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha \end{pmatrix} \notin S \quad \text{if} \quad \alpha \neq 1.$$

$$\begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} x+y \\ 2 \end{pmatrix} \notin S.$$

So both closure properties ~~hold~~ fail to hold. Thus S is not a subspace. (Note: Only 1 needs to fail to get a negative answer.)

This is actually the same example we did last class.



Recall that the proper subspaces of $\mathbb{R}^2$ were the lines that passed through the origin.

Ex. $S = \{ A \in \mathbb{R}^{2\times 2} \mid a_{12} = -a_{21} \}$

Verify that this is a subspace. (of $\mathbb{R}^{2\times 2}$).

Ex. Let $S = \{ p \in P_n \mid p(0) = 0 \}$.

Show that S is a subspace of P_n.

RE Ex. Let $C^n[a,b]$ be the set of all functions f in $C[a,b]$ that have a continuous n^{th} order derivative $f^{(n)} \in C(a,b)$.

Verify that $C^n[a,b]$ is a subspace.

Ex. The function $f(x) = |x|$ is continuous on $[-1,1]$, but it is not differentiable. Therefore $f \notin C'[-1,1]$, and $C'[-1,1]$ is a proper subspace of $C[-1,1]$.

(RE) Show that $g(x) = x|x|$ is in $C[-1,1]$ and $C'[-1,1]$, but not $C^2[-1,1]$. Therefore C^2 is a proper subspace of both C and C'.

Ex. $S = \{ f \in C^2[a,b] \mid f''(x) + f(x) = 0,\ x \in [a,b] \}$.

(RE) Verify that S is a subspace of $C^2[a,b]$.

This means that the set of all solutions on $[a,b]$ of the DE $y'' + y = 0$ forms a vector space.

(RE) Verify that $y_1(x) = \sin(x)$ and $y_2(x) = \cos(x)$ are both in S.

This means that any function of the form

$$f(x) = c_1 \sin(x) + c_2 \cos(x)$$

is in S; hence also a solution of $y'' + y = 0$.

<u>The Null Space of a Matrix</u>

Let $A \in \mathbb{R}^{m \times n}$, and let $N(A)$ be the set of all solutions of the homogeneous system $A\underline{x} = \underline{0}$.

$$N(A) = \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0} \}.$$

This set is called the <u>null space</u> of A, and always forms a vector subspace of $\mathbb{R}^n$.

Let $\underline{x} \in N(A)$, $\underline{y} \in N(A)$, and $\alpha \in \mathbb{R}$.

C1. $A(\alpha \underline{x}) = \alpha A(\underline{x}) = \alpha \cdot \underline{0} = \underline{0}$. so $\alpha \underline{x} \in N(A)$

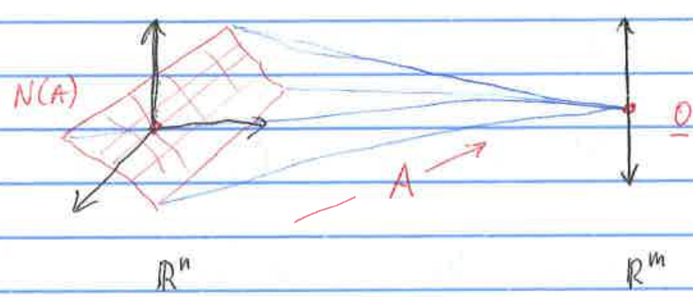
C2. $A(\underline{x}+\underline{y}) = A(\underline{x}) + A(\underline{y}) = \underline{0} + \underline{0} = \underline{0}$ so $\underline{x}+\underline{y} \in N(A)$.

Notice that $\underline{0} \in N(A)$ since the system is homogeneous.

<u>Idea:</u> Think of $A \in \mathbb{R}^{m\times n}$ as a function that takes in a vector $\underline{x} \in \mathbb{R}^n$ and gives back a vector $\underline{b} \in \mathbb{R}^m$.

$$\mathbb{R}^n \xrightarrow{\ A\ } \mathbb{R}^m$$
$$\underline{x} \longmapsto \underline{b}$$

The null space of A is the set in $\mathbb{R}^n$ that gets sent to the $\underline{0}$ vector in $\mathbb{R}^m$:



We can't draw higher dimensions, but the schematic picture always looks "like this."

<u>Ex.</u> Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Find $N(A)$.

Use RREF to get $N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\}$

The Span of a Set of Vectors

Defn. Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n \in V$. A sum of the form

$$\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \cdots + \alpha_n \underline{x}_n \quad , \alpha_i \in \mathbb{R}$$

is called a linear combination of $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$.
The set of all linear combinations of $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ is called the span of $\underline{x}_1, \ldots, \underline{x}_n$, and we write $\mathrm{Span}\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$.

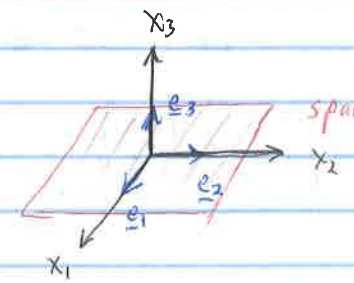
In this lingo, we just found that the null space of A was the span of $(1,-2,1,0)^T$ and $(-1,1,0,1)^T$:

$$N(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Ex. Let $\underline{e}_1, \underline{e}_2 \in \mathbb{R}^3$. The span of $\underline{e}_1$ and $\underline{e}_2$ is

$$\mathrm{Span}\{\underline{e}_1, \underline{e}_2\} = \{ \alpha \underline{e}_1 + \beta \underline{e}_2 \mid \alpha, \beta \in \mathbb{R} \}$$

or the set of all vectors that look like $(\alpha, \beta, 0)^T$.



span $\{\underline{e}_1, \underline{e}_2\}$ is the "xy plane" in 3-space.

Thm. Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ be elements of a vector space V. Then $\mathrm{Span}\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$ is a subspace of V.

Proof. VERY STRONGLY RECOMMENDED EXERCISE!
It's in the book on p. 122, so the RE is to understand it and be able to recreate it in your sleep.

Spanning Sets for a Vector Space

Def'n. The set $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is a spanning set for V if and only if every vector in V can be written as a linear combination of $\underline{v}_1, \dots, \underline{v}_n$.

i.e., iff $\operatorname{Span}\{\underline{v}_1, \dots, \underline{v}_n\} = V$.

Ex. Which of the following are spanning sets for $\mathbb{R}^3$?

a) $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, (1,2,3)^T\}$ — Yes, obvious.
b) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$ — Yes
c) $\{(1,0,1)^T, (0,1,0)^T\}$ — No
d) $\{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$ — No.

How do we determine this?
Take b) for example.
We ask:
For any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ can we find x_1, x_2, x_3 such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In other words, is the system $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ consistent for any choice of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$?

We know numerous ways of determining this. If the vectors form a square matrix, then just take the determinant. In this case

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \pm 1 \neq 0 \quad (-1?)$$

Ex. The vectors $1-x^2$, $x+2$, x^2 span P_3. Thus if ax^2+bx+c is a polynomial in P_3, then it is possible to find scalars α_1, α_2, and α_3 such that

$$ax^2+bx+c = \alpha_1(1-x^2) + \alpha_2(x+2) + \alpha_3 x^2$$

RE. Solve for $\alpha_1, \alpha_2, \alpha_3$ in terms of a, b, c.

Problem 3.2.5 on p. 125 is in the Good Problems 1 packet. Be careful doing this one. :)