Chapter 2: Determinants

2.1: The Determinant of a Matrix

Let $A \in \mathbb{R}^{n\times n}$. It is possible to associate a real number $\det(A) \in \mathbb{R}$ to A that tells us if A is singular or not.

[$\det(A)$ also has some geometric significance that we will see later.]

Case I. $A \in \mathbb{R}^{1\times 1} = \mathbb{R}$. Then $\det(A) = A$.

Case II. Let $A \in \mathbb{R}^{2\times 2}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

By a previous thm, A is nonsingular iff it is row equivalent to $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus we may assume that $a \neq 0$.
Multiply a times R_2:

$$A \to \begin{pmatrix} a & b \\ ac & ad \end{pmatrix}$$

Subtract cR_1 from R_2

$$\to \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}$$

Since $a \neq 0$, this matrix is row equiv. to I iff $ad-bc \neq 0$.

We define the determinant to be $\det(A) = ad-bc \in \mathbb{R}$.
A is nonsingular if and only if $\det(A) \neq 0$.

Notation. $\det(A) = |A|$

Ex. $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ $\det(A) = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3(1) - 4(2) = 3 - 8 = -5 \neq 0.$

Case III. 3×3 matrices become more involved.

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \longrightarrow \begin{pmatrix} a & b & c \\ 0 & \frac{ae-bd}{a} & \frac{af-bc}{a} \\ 0 & \frac{ah-gb}{a} & \frac{ak-gc}{a} \end{pmatrix}$

This will be row equiv. to I iff

$$a \begin{vmatrix} \frac{ae-bd}{a} & \frac{af-bc}{a} \\ \frac{ah-gb}{a} & \frac{ak-gc}{a} \end{vmatrix} \neq 0$$

This can be simplified (w/ some "messy" algebra) to:

$(ae-bd)(ak-gc) - (ah-gb)(af-bc) \neq 0$

$$a^2ek - aegc - abdk + bcdg - (a^2hf - abch - abgf + b^2gc) \neq 0$$

*! $\boxed{\text{This notation is bad!}}$

* Maybe skip all of this?

W/ better notation, we may define

$$\boxed{\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ & + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}. \end{aligned}}$$

This is messy looking and hard to memorize. A method would be better!

<u>Defn.</u> Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, and let M_{ij} denote the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A. The determinant of M_{ij} is called the <u>(i,j)-th</u> <u>minor</u> <u>of</u> <u>A</u>, or the <u>minor</u> <u>of</u> <u>a_{ij}</u>.

The <u>cofactor</u> <u>of</u> <u>a_{ij}</u> (or the (i,j)-th cofactor of A) is the number $A_{ij} = (-1)^{i+j}\det(M_{ij})$.

For $A \in \mathbb{R}^{2\times 2}$, $\det(A) = a_{11}A_{11} + a_{12}A_{12}$. (<u>RE</u>: verify it!)

This is called the <u>cofactor</u> <u>expansion</u> of $\det(A)$.

<u>Defn.</u> Let $A \in \mathbb{R}^{n\times n}$. The <u>determinant</u> of A is the number $\det(A) \in \mathbb{R}$ given by:

$$\det(A) = \begin{cases} A = a_{11} & \text{if } n=1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n>1 \end{cases}$$

where $A_{1j} = (-1)^{1+j}\det(M_{1j})$ $j=1,\dots,n$.

<u>Ex.</u> $A = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$ $\det(A) = -16$

* It turns out we can expand along any column or row.

<u>Ex.</u> Expand on the second row of last <u>Ex</u>.

Ex. $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} = 2\cdot(-1)\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 2\cdot(-1)\cdot 3\cdot(+1)\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

$= -6\,(10-12) = 12$ □

Thm. If $A \in \mathbb{R}^{n\times n}$, then $\det(A^T) = \det(A)$.

Proof. Expanding on the i^{th} column of A^T is equivalent to expanding on the i^{th} row of A. □

Details in the text, p. 90.

Thm. If $A \in \mathbb{R}^{n\times n}$ is triangular, then $\det(A)$ equals the product of the diagonal elements.

Proof. RE # 8. □

Thm. Let $A \in \mathbb{R}^{n\times n}$.

1. If A has a row or column of all zeros, then $\det(A) = 0$.
2. If A has two identical rows or columns, then $\det(A) = 0$.