# Math 511: Linear Algebra
# Good Problems 8

Due: Thursday, 24 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

**Name:** Key

**Instructions:** Complete all 5 problems. Each problem is worth 20 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

**1.** Let $A \in \mathbb{R}^{n\times n}$. Show that $A$ and $A^T$ have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

$$\begin{aligned}\det(A-\lambda I) &= \det\left((A-\lambda I)^T\right)\\ &= \det\left(A^T-(\lambda I)^T\right)\\ &= \det\left(A^T-\lambda I^T\right)\\ &= \det\left(A^T-\lambda I\right). \qquad \square\end{aligned}$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, shift left, "A"] & \mathbb{R}^n \arrow[l, shift left, "A^T"] \end{tikzcd}$$

They map in different directions, so they do not have the same e'vectors in general.

Recall $R(A^T) = N(A)^{\perp}$.

**2.** A matrix $Q \in \mathbb{R}^{n \times n}$ is said to be *orthogonal* if $Q^T Q = I$.

(*a.*) Show that the if $\lambda$ is an eigenvalue of $Q$, then $|\lambda| = 1$.

(*b.*) Show that $|\det(Q)| = 1$.

a.) Since $Q^T Q = I$, then $Q^T Q \underline{x} = \underline{x}$ for all $\underline{x} \in \mathbb{R}^n$.

In particular, if $\underline{x}$ is an evector for some e'value $\lambda$ of $Q$

then $\underline{x} = Q^T Q \underline{x} = Q^T(\lambda \underline{x}) = \lambda(Q^T \underline{x})$

For this to equal $\underline{x}$, $Q^T \underline{x}$ must equal $\frac{1}{\lambda}\underline{x}$.

But by the last problem, the e'values of $Q$ = e'values of $Q^T$.

Thus $\lambda = \frac{1}{\lambda} \Rightarrow \lambda^2 = 1 \Rightarrow |\lambda| = 1$.

b.) $\det Q = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \pm 1$ if $\lambda_i = \pm 1$ for all $i$.

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**3.** Solve the $3^{rd}$-order initial value problem

$$\begin{cases} y''' = -4y + 4y' - y''; \\ y(0) = y'(0) = 1, \\ y''(0) = -1. \end{cases}$$

Put $Y = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$ so $Y' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & -1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = AY$

$p(\lambda) = \det(A - \lambda I) = -\lambda^3 - \lambda^2 + 4\lambda - 4$

e'values are

$\lambda_1 \approx -2.8751$

$\lambda_2, \lambda_3 \approx 0.9376 \pm 0.7157 i$

Stop here since e'vectors will be nasty!

In general

$Y = c_1 e^{\lambda_1 t} \underline{x}_1 + c_2 e^{\lambda_2 t} \underline{x}_2 + c_3 e^{\lambda_3 t} \underline{x}_3$

where $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = X^{-1} Y(0)$, $X = (\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

On exam, e'values will be "nice".

**4.** Factor the matrix $A$ into a product $XDX^{-1}$ where $D$ is diagonal.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

Using Wolfram: (You need to do this by hand...)

$\lambda_1 = -2$ $\quad \underline{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\quad \underline{x}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ $\quad \underline{x}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$

$\lambda_3 = 1$

so $X = \begin{pmatrix} 0 & 0 & 3 \\ -1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and $X^{-1} = \frac{1}{12}\begin{pmatrix} -4 & -3 & 9 \\ 0 & 3 & 3 \\ 4 & 0 & 0 \end{pmatrix}$

so $A = \frac{1}{12}\begin{pmatrix} 0 & 0 & 3 \\ -1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -4 & -3 & 9 \\ 0 & 3 & 3 \\ 4 & 0 & 0 \end{pmatrix}$.

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**5.** Solve the initial value problem $\mathbf{Y}' = A\mathbf{Y}$, $\mathbf{Y}_0 = (1,1,-1)^T$ by computing $e^{tA}\mathbf{Y}_0$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$A^3 = 0.$$

So $$e^{tA} = I + tA + \tfrac{1}{2}t^2 A^2 + 0$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & t \\ t & 0 & t \\ -t & -t & -t \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t^2 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 \\ -\frac{1}{2}t^2 & 0 & -\frac{1}{2}t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t+\frac{1}{2}t^2 & t & t+\frac{1}{2}t^2 \\ t & 1 & t \\ -t-\frac{1}{2}t^2 & -t & 1-t-\frac{1}{2}t^2 \end{pmatrix}$$

and $$Y = e^{tA}Y_0 = \begin{pmatrix} 1+t+\cancel{\frac{1}{2}t^2}+\cancel{t}-\cancel{t}-\cancel{\frac{1}{2}t^2} \\ \cancel{t}+1-\cancel{t} \\ -t-\cancel{\frac{1}{2}t^2}-\cancel{t}-1+\cancel{t}+\cancel{\frac{1}{2}t^2} \end{pmatrix}$$

$$\boxed{Y = \begin{pmatrix} 1+t \\ 1 \\ -1-t \end{pmatrix}}$$