Math 511: Linear Algebra
Good Problems 7

Due: Friday, 18 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 5 problems. Each problem is worth 20 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Given the ordered basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ for $\mathbb{R}^3$ where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

use the Gram-Schmidt process to obtain an orthonormal basis.

$$\underline{u}_1 = \frac{\underline{x}_1}{\|\underline{x}_1\|} = \frac{1}{\sqrt{9}}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} = \underline{u}_1}$$

$$\underline{p}_1 = \langle \underline{x}_2, \underline{u}_1 \rangle \underline{u}_1 = \left(\frac{4}{3} + \frac{6}{3} - \frac{4}{3}\right)\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

$$\underline{x}_2 - \underline{p}_1 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 4/3 \\ -4/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 5/3 \\ 10/3 \end{pmatrix} \approx \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\underline{u}_2 = \frac{\underline{x}_2 - \underline{p}_1}{\|\underline{x}_2 - \underline{p}_1\|} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \underline{u}_2}$$

So $U = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$

$$\underline{p}_2 = \langle \underline{x}_3, \underline{u}_1 \rangle \underline{u}_1 + \langle \underline{x}_3, \underline{u}_2 \rangle \underline{u}_2$$

$$= \left(\frac{1}{3} + \frac{4}{3} - \frac{2}{3}\right)\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} + \left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right)\begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} 4/3 \\ 2/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 4/3 \\ 2/3 \end{pmatrix}$$

$$\underline{x}_3 - \underline{p}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 5/3 \\ 4/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix} \approx \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\underline{u}_3 = \frac{\underline{x}_3 - \underline{p}_2}{\|\underline{x}_3 - \underline{p}_2\|} = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \underline{u}_3}$$

2. The vectors

$$\mathbf{x}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_2 = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of X^T, where $X = (2\mathbf{x}_1, 6\mathbf{x}_2)$. [Hint: find a basis for $N(X^T)$ first, then use Gram-Schmidt.]

$$X^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

so $N(X^T)$:

$$x_1 = -\beta + \alpha, \quad x_2 = \beta, \quad x_3 = -2\alpha, \quad x_4 = \alpha$$

$$N(X^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\underline{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \underline{u}_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{p}_3 = \langle \underline{x}_4, \underline{u}_3 \rangle \underline{u}_3 = \left(\frac{-1}{\sqrt{2}}\right) \begin{pmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{x}_4 - \underline{p}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$$\underline{u}_4 = \frac{\underline{x}_4 - \underline{p}_3}{\|\underline{x}_4 - \underline{p}_3\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 1 \\ 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{22} \\ 1/\sqrt{22} \\ -4/\sqrt{22} \\ 2/\sqrt{22} \end{pmatrix}$$

O-N basis for $\mathbb{R}^4$ is:

$$\boxed{\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{22}}\begin{pmatrix} 1 \\ 1 \\ -4 \\ 2 \end{pmatrix} \right\}}$$

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3. Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

$$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (-2-\lambda)\left[(-\lambda)(-1-\lambda) + 1\right] + 1$$

$$= -(\lambda+2)\left[\lambda^2 + \lambda + 1\right] + 1$$

$$= -\left(\lambda^3 + \lambda^2 + \lambda + 2\lambda^2 + 2\lambda + 2\right) + 1$$

$$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1$$

$$= -(\lambda+1)^3$$

so $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = -1}$

$N(A+I)$: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

$x_1 = \alpha$
$x_2 = 0$
$x_3 = \alpha$

so $\boxed{N(A+I) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}.}$

4. (*a.*) Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *idempotent* if $A^2 = A$. Show that if A is idempotent then each of its eigenvalues is either 1 or 0.

Let λ be an e'value w/ corresponding evector $\underline{x}$.

Then $A\underline{x} = \lambda \underline{x}$

and $A^2\underline{x} = A(A\underline{x}) = A(\lambda\underline{x}) = \lambda(A\underline{x}) = \lambda(\lambda\underline{x}) = \lambda^2\underline{x}$.

Thus $A^2 = A \Rightarrow \lambda^2\underline{x} = \lambda\underline{x} \Rightarrow (\lambda^2-\lambda)\underline{x} = \underline{0}$.

Since $\underline{x}$ is an e'vector, $\underline{x} \neq \underline{0}$. Thus $\lambda^2-\lambda$ must $=0$.

$\lambda^2-\lambda = \lambda(\lambda-1) = 0$

has roots $\lambda = 0$ or $\lambda = 1$. $\square$

(*b.*) A matrix is called *nilpotent* if $A^k = 0$ for some $k \in \mathbb{Z}^+$. Show that if A is nilpotent, then all of its eigenvalues are 0.

Let λ be an e'value w/ corresponding e'vector $\underline{x}$.

Then. $A\underline{x} = \lambda\underline{x}$ and $A^k\underline{x} = \lambda^k\underline{x}$ by argument above.

But $A^k\underline{x} = \underline{0}$, so $\lambda^k\underline{x} = \underline{0}$.

Again $\underline{x} \neq \underline{0} \Rightarrow \lambda^k = 0 \Rightarrow \lambda = 0$. $\square$

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5. Solve the initial value problem

$$\begin{cases} y_1' &= y_1 - 2y_2, \\ y_2' &= 2y_1 + y_2; \\ y_1(0) &= 1, \\ y_2(0) &= -2. \end{cases}$$

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$

$$Y' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} Y$$

e'values of A: $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + 4 = 0$

$$\lambda - 1 = \pm 2i$$
$$\lambda = 1 \pm 2i$$

$N(A-(1+2i)I)$: $\begin{pmatrix} 1-1-2i & -2 \\ 2 & 1-1-2i \end{pmatrix} = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \to \left(\begin{array}{cc|c} 2 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = i\alpha$
$x_2 = \alpha$ $\quad$ so $\underline{x} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

$N(A-(1-2i)I)$: $\left(\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = \alpha - i$, $x_2 = \alpha$

$\to \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad$ so $\underline{x} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

Thus $X = \begin{pmatrix} 1+i & 1-i \\ 1 & 1 \end{pmatrix}$

Thus $Y(t) = C_1 e^{(1+2i)t}\begin{pmatrix}1+i\\1\end{pmatrix} + C_2 e^{(1-2i)t}\begin{pmatrix}1-i\\1\end{pmatrix}$

$$= C_1\left(e^t\cos 2t + ie^t \sin 2t\right)\begin{pmatrix}1+i\\1\end{pmatrix} + C_2\left(e^t\cos 2t - ie^t\sin 2t\right)\begin{pmatrix}1-i\\1\end{pmatrix}$$

$$= \begin{pmatrix} C_1e^t\cos 2t + iC_1e^t\sin 2t + C_2e^t\cos 2t - C_2 ie^t\sin 2t \\ + C_1ie^t\cos 2t - C_1e^t\sin 2t - iC_2e^t\cos 2t + C_2e^t\sin 2t \\ C_1e^t\cos 2t + iC_1e^t\sin 2t + C_2e^t\cos 2t - iC_2e^t\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} C_1e^t\cos 2t + C_2e^t\cos 2t - C_1e^t\sin 2t + C_2e^t\sin 2t \\ C_1e^t\cos 2t + C_2e^t\cos 2t \end{pmatrix}$$

$$+ i\begin{pmatrix} C_1e^t\sin 2t - C_2e^t\sin 2t + C_1e^t\cos 2t - C_2e^t\cos 2t \\ C_1e^t\sin 2t - C_2e^t\sin 2t \end{pmatrix}$$

Put $A = C_1 + C_2 \quad B = C_1 - C_2$

$$= \begin{pmatrix} Ae^t\cos 2t - Be^t\sin 2t \\ Ae^t\cos 2t \end{pmatrix} + i\begin{pmatrix} Be^t\sin 2t + Be^t\cos 2t \\ Be^t\sin 2t \end{pmatrix}$$

Now, $Y(0) = \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}A\\A\end{pmatrix} + i\begin{pmatrix}B\\0\end{pmatrix}$

So $A + iB = 1 \Rightarrow iB = 1+2 = 3 \Rightarrow B = -3i$

and $-2 = A$

Thus,

$$Y(t) = \begin{pmatrix} -2e^t\cos 2t + 3ie^t\sin 2t + 3e^t\sin 2t + 3e^t\cos 2t \\ \\ -2e^t\cos 2t + 3e^t\sin 2t \end{pmatrix}$$

?!

$$= \begin{pmatrix} e^t\cos 2t + (3+3i)e^t\sin(2t) \\ \\ -2e^t\cos 2t + 3e^t\sin 2t \end{pmatrix}$$