Math 511: Linear Algebra
Good Problems 6

Due: Thursday, 11 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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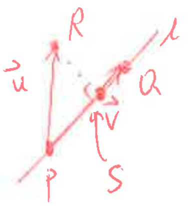
1. (*a.*) Find the point on the line $y = 2x + 1$ that is closest to the point $(5,2)$.

P: $x=0$: $(0,1)$ } on the line, so $\vec{v} = \overrightarrow{PQ} = \begin{pmatrix}1\\2\end{pmatrix}$ is "on" the line.

Q: $x=1$: $(1,3)$

R: $(5,2)$ so $\vec{u} = \overrightarrow{PR} = \begin{pmatrix}5\\1\end{pmatrix}$ is ~~the~~ a vector from the line to R.



$$\underline{p} = \mathrm{proj}_{\vec{v}}\vec{u} = \frac{\langle \vec{u},\vec{v}\rangle}{\langle \vec{v},\vec{v}\rangle}\vec{v} = \frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}7/5\\14/5\end{pmatrix}$$

$$\begin{pmatrix}7/5\\14/5\end{pmatrix} = \begin{pmatrix}s_1\\s_2\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}s_1\\s_2\end{pmatrix} = \boxed{\begin{pmatrix}7/5\\19/5\end{pmatrix} = S}$$

(*b.*) Find the distance from the point $(1,2)$ to the line $4x - 3y = 0$.

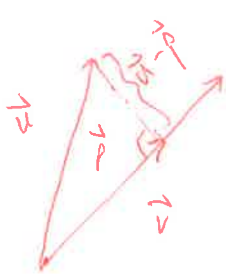
O : $(0,0)$ is on the line.

$(3,4)$ is also on the line

put $\vec{u} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\underline{p} = \mathrm{proj}_{\vec{v}}\vec{u} = \frac{11}{25}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}33/25\\44/25\end{pmatrix}$

$\vec{v} = \begin{pmatrix}3\\4\end{pmatrix}$

$$\underline{u}-\underline{p} = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}33/25\\44/25\end{pmatrix} = \begin{pmatrix}-8/25\\6/25\end{pmatrix}$$



$$\|u-p\| = \sqrt{\frac{64+36}{625}} = \sqrt{\frac{100}{625}} = \frac{10}{25} = \boxed{\frac{2}{5}} \; !$$

2. Show that if **u** and **v** are any vectors in $\mathbb{R}^2$, then $\|\mathbf{u}+\mathbf{v}\|^2 \le (\|\mathbf{u}\|+\|\mathbf{v}\|)^2$, and hence $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$. When does equality hold? Give a geometric interpretation of the inequality.

Pf. $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$$\le \left| \|u\|^2 + 2\langle u,v\rangle + \|v\|^2 \right|$$

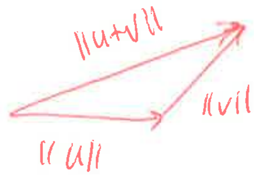
$$= \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$$

$$\le \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\| \quad (\text{by CSB} \neq)$$

$$= (\|u\| + \|v\|)^2$$

$\square$

This is the triangle inequality!



$\|u+v\| \le \|u\| + \|v\|$!

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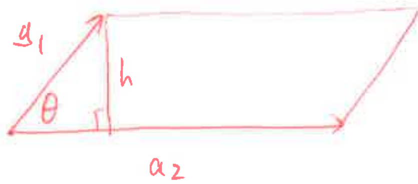
3. Let $A \in \mathbb{R}^{2\times 2}$ with linearly independent column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are used to form a parallelogram P with altitude h (see the figure on page 213 of the textbook), show that

1. $h^2\|\mathbf{a}_2\|^2 = \|\mathbf{a}_1\|^2\|\mathbf{a}_2\|^2 - \langle \mathbf{a}_1, \mathbf{a}_2\rangle^2$, and
2. Area of $P = |\det(A)|$.

$$A = (\underline{a}_1, \underline{a}_2)$$

$$A = \begin{pmatrix} \underbrace{\begin{matrix} a_{11} \\ a_{21} \end{matrix}}_{\underline{a}_1} & \underbrace{\begin{matrix} a_{12} \\ a_{22} \end{matrix}}_{\underline{a}_2} \end{pmatrix}$$



$$\cos^2\theta = \frac{\langle \underline{a}_1, \underline{a}_2\rangle^2}{\|\underline{a}_1\|^2\|\underline{a}_2\|^2} \quad \text{so} \quad \sin^2\theta = 1 - \cos^2\theta = \frac{\|\underline{a}_1\|^2\|\underline{a}_2\|^2 - \langle \underline{a}_1, \underline{a}_2\rangle^2}{\|\underline{a}_1\|^2\|\underline{a}_2\|^2}$$

but $\sin^2\theta = \dfrac{h^2}{\|\underline{a}_1\|^2}$ also.

Thus, $$\frac{h^2}{\|\underline{a}_1\|^2} = \frac{\|\underline{a}_1\|^2\|\underline{a}_2\|^2 - \langle \underline{a}_1, \underline{a}_2\rangle^2}{\|\underline{a}_1\|^2\|\underline{a}_2\|^2} \Rightarrow \boxed{h^2\|\underline{a}_2\|^2 = \|\underline{a}_1\|^2\|\underline{a}_2\|^2 - \langle \underline{a}_1, \underline{a}_2\rangle^2}$$

$$\text{Area} = h\|\underline{a}_2\| = \sqrt{\|\underline{a}_1\|^2\|\underline{a}_2\|^2 - \langle \underline{a}_1, \underline{a}_2\rangle^2} = \sqrt{(a_{11}^2 + a_{21}^2)(a_{12}^2 + a_{22}^2) - (a_{11}a_{12} + a_{21}a_{22})^2}$$

$$= \sqrt{\cancel{a_{11}^2a_{12}^2} + a_{11}^2a_{22}^2 + a_{21}^2a_{12}^2 + \cancel{a_{21}^2a_{22}^2} - \cancel{a_{11}^2a_{12}^2} - \cancel{a_{21}^2a_{22}^2} - 2a_{11}a_{12}a_{21}a_{22}}$$

$$= \sqrt{a_{11}^2a_{22}^2 - 2a_{11}a_{12}a_{21}a_{22} + a_{21}^2a_{12}^2}$$

$$= \sqrt{(a_{11}a_{22} - a_{21}a_{12})^2}$$

$$= |a_{11}a_{22} - a_{21}a_{12}|$$

$$= |\det A|. \quad \square$$

4. Consider the vectors in $\mathbb{R}^4$,

$$\mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ -4 \\ 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

Determine the angle between **x** and **y**. Determine the distance between **x** and **y**.

$$\|x\| = \sqrt{4 \cdot 4^2} = \sqrt{8^2} = 8$$

$$\|y\| = \sqrt{4^2 + 2 \cdot 2^2 + 1} = \sqrt{25} = 5$$

$$\langle x, y \rangle = 16 + 8 - 8 + 4 = 20$$

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|\|y\|} = \frac{20}{8 \cdot 5} = \frac{20}{40} = \frac{1}{2} \quad \text{so} \quad \boxed{\theta = \frac{\pi}{3}}$$

$$x - y = \begin{pmatrix} 0 \\ 2 \\ -6 \\ 3 \end{pmatrix}$$

$$\|x - y\| = \sqrt{0 + 4 + 36 + 9} = \sqrt{49} = \boxed{7.}$$

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5. Consider the vectors in $\mathbb{R}^4$,

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 5 \\ -3 \end{pmatrix}.$$

Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{x}+\mathbf{y}\|_2$, and verify that the Pythagorean theorem holds.

$\langle x, y \rangle = -1\cdot 1 + (-1)\cdot 1 + 1\cdot 5 + 1(-3) = -1 - 1 + 5 - 3 = 0$

so $x \perp y$.

$\|x\|_2 = \sqrt{(-1)^2 + (-1)^2 + 1^2 + 1^2} = \sqrt{4} = 2$.

$\|y\|_2 = \sqrt{1^2 + 1^2 + 5^2 + (-3)^2} = \sqrt{1+1+25+9} = \sqrt{36} = 6$

$x + y = \begin{pmatrix} 0 \\ 0 \\ 6 \\ -2 \end{pmatrix}$

$\|x+y\|_2 = \sqrt{6^2 + 2^2} = \sqrt{40}$

Pyth. Thm.

$\|x\|_2^2 + \|y\|_2^2 = 2^2 + 6^2 = 4 + 36 = 40$

$\|x+y\|_2^2 = (\sqrt{40})^2 = 40$ ✓ !

6. In $C[0,1]$, define an inner product by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx. \qquad (*)$$

1. Verify that this is indeed an inner product.
2. Compute the following inner products: $\langle e^x, e^{-x}\rangle$, $\langle x, \sin(\pi x)\rangle$, $\langle x^2, x^3\rangle$.

1.)

1.) $\langle f,f\rangle = \int_0^1 f(x)f(x)\,dx = \int_0^1 (f(x))^2 dx \geq 0$ since $(f(x))^2 \geq 0$,

$= 0$ iff $(f(x))^2 = 0$ for all x,

hence iff $f = 0$.

2.) $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g,f\rangle$.

3.) $\langle \alpha f+\beta g, h\rangle = \int_0^1 (\alpha f(x)+\beta g(x))h(x)\,dx = \alpha\int_0^1 f(x)h(x) + \beta\int_0^1 g(x)h(x)\,dx$

$= \alpha\langle f,h\rangle + \beta\langle g,h\rangle$. $\square$

2.) $\langle e^x, e^{-x}\rangle = \int_0^1 e^x e^{-x}\,dx = \int_0^1 1\,dx = 1$.

$\langle x, \sin(\pi x)\rangle = \int_0^1 x\sin(\pi x)\,dx = \frac{-x\cos\pi x}{\pi}\Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos(\pi x)\,dx$

$= \frac{-1}{\pi}\cos(\pi) - 0 + \underbrace{\frac{1}{\pi^2}\sin(\pi)}_{0} - \underbrace{\frac{1}{\pi^2}\sin(0)}_{0}$

$= \boxed{\frac{1}{\pi}}$

$\langle x^2, x^3\rangle = \int_0^1 x^5\,dx = \frac{1}{6}x^6\Big|_0^1 = \boxed{\frac{1}{6}}$.

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7. Using the inner product on $C[0,1]$ defined by equation $(*)$, (*a.*) find the angle θ between the vectors 1 and x; (*b.*) then find the vector projection $\mathbf{p}$ of 1 onto x, and verify that $\mathbf{1}-\mathbf{p}$ is orthogonal to $\mathbf{p}$; (*c.*) Compute $\|1-\mathbf{p}\|$, $\|\mathbf{p}\|$, and $\|1\|$, and verify that the Pythagorean theorem holds.

$\|1\|^2 = \langle 1,1\rangle = \int_0^1 1\,dx = 1$ so $\|1\| = 1$.

$\|x\|^2 = \langle x,x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$ so $\|x\| = \frac{1}{\sqrt{3}}$.

$\langle 1,x\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

so $\cos\theta = \dfrac{1/2}{1/\sqrt{3}} = \dfrac{\sqrt{3}}{2} \Rightarrow \boxed{\theta = \frac{\pi}{6}}$ a.)

$\underline{p} = \mathrm{Proj}_x 1 = \dfrac{\langle 1,x\rangle}{\langle x,x\rangle}x = \dfrac{1/2}{1/3}x = \boxed{\frac{3}{2}x}$ b)

$1-p = 1-\frac{3}{2}x$

$\left\langle 1-\frac{3}{2}x, \frac{3}{2}x\right\rangle = \int_0^1 \frac{3}{2}x - \frac{9}{4}x^2\,dx = \frac{3}{4}x^2 - \frac{3}{4}x^3\Big|_0^1 = \frac{3}{4}-\frac{3}{4}-0 = 0.$

so $p \perp (1-p)$!

$\|p\|^2 = \int_0^1 \frac{9}{4}x^2\,dx = \frac{3}{4}x^3\Big|_0^1 = \frac{3}{4}$

$\|1-p\|^2 = \int_0^1 1-3x+\frac{9}{4}x^2\,dx = x - \frac{3}{2}x^2 + \frac{3}{4}x^3\Big|_0^1 = 1 - \frac{3}{2} + \frac{3}{4} = \frac{1}{4}$

$\|p\|^2 + \|1-p\|^2 = \frac{3}{4} + \frac{1}{4} = 1 = \|1\|^2 = \|p+1-p\|^2$!

8. Prove that, for any $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V,

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\,\|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|^2.$$

Give a geometric interpretation of this result for the vector space $\mathbb{R}^2$.

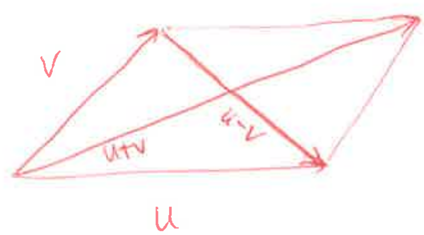
$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

Similarly, $\|u-v\|^2 = \|u\|^2 - 2\langle u,v\rangle + \|v\|^2$

Adding these,

$$\boxed{\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.} \qquad \square$$

Geometrically.



This is the parallelogram "Law"!

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9. The result of problem 8 is not valid for norms that are not derived from an inner product. Show that the result does not hold for the norm $\|\cdot\|_1$ on $\mathbb{R}^2$.

Consider $u = \begin{pmatrix}1\\1\end{pmatrix}$ and $v = \begin{pmatrix}1\\0\end{pmatrix}$

Then $u+v = \begin{pmatrix}2\\1\end{pmatrix}$ and $u-v = \begin{pmatrix}0\\1\end{pmatrix}$

$\|u+v\|_1^2 = (|2|+|1|)^2 = 3^2 = 9$

$+\ \|u-v\|_1^2 = |0|+|1| = 1^2 = 1$

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but

$\|u\|_1^2 = (1+1)^2 = 2^2 = 4$

$\|v\|_1^2 = (1+0)^2 = 1^2 = 1$

let $u = \underline{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $v = \underline{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$\underline{e}_1 + \underline{e}_2 = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad \underline{e}_1 - \underline{e}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$\|\underline{e}_1 + \underline{e}_2\|_1^2 = (1+1)^2 = 2^2 = 4$

$+\ \|\underline{e}_1 - \underline{e}_2\|_1^2 = (1+1)^2 = 2^2 = 4$

8

$\|\underline{e}_1\|_1^2 = 1^2 = 1$

$\|\underline{e}_2\|_1^2 = 1^2 = 1$

so $2\|e_1\|_1^2 + 2\|e_2\|_1^2 = 2+2 = 4$

$4 \neq 8$!

10. Show that if **u** and **v** are vectors in an inner product space V that satisfy the Pythagorean "law",

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2,$$

then **u** and **v** must be orthogonal.

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \Rightarrow \|u+v\|^2 - \|u\|^2 - \|v\|^2 = 0.$$

$$\|u+v\|^2 = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$$\Rightarrow \langle u,v\rangle = \frac{1}{2}\left(\underbrace{\|u+v\|^2 - \|u\|^2 - \|v\|^2}_{0}\right)$$

$$\text{so } \langle u,v\rangle = 0 \Rightarrow u \perp v.$$