Math 511: Linear Algebra
Good Problems 5

Due: Thursday, 3 July 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Let L be a linear operator on $\mathbb{R}^1$ and let $a = L(1)$. Show that $L(x) = ax$ for all $x \in \mathbb{R}^1$.

Notice that $\{1\}$ is a basis for $\mathbb{R}^1$.
Any $x \in \mathbb{R}^1$ can be written as $x \cdot 1$.

Then $L(x) = L(x \cdot 1) = x L(1) = x \cdot a$

so $L(x) = ax \quad \forall x \in \mathbb{R}$.

2. Let L be a linear operator on a vector space V. Define L^n, $n \geq 1$, recursively by

$$L^1 = L,$$
$$L^{k+1}(\mathbf{v}) = L(L^k(\mathbf{v})),$$

for all $\mathbf{v} \in V$. Show that L^n is a linear operator on V for each $n \geq 1$. [Hint: Use mathematical induction.]

If A is a matrix representation of L, what is the matrix representation of L^n? Justify your answer.

$L^1 = L$ is linear by assumption.

Induction hypothesis: Assume L^k is linear for some $k \geq 1$.

(L1.) $L^{k+1}(\alpha \underline{x}) = L(L^k(\alpha \underline{x})) = L(\alpha L^k(\underline{x}))$ by the induction hyp.

$= \alpha L(L^k(\underline{x}))$ since L is linear

$= \alpha L^{k+1}(\underline{x})$.

(L2.) $L^{k+1}(\underline{x}+\underline{y}) = L(L^k(\underline{x}+\underline{y})) = L(L^k(\underline{x}) + L^k(\underline{y}))$ by ind. hyp.

$= L(L^k(\underline{x})) + L(L^k(\underline{y}))$ since L is linear

$= L^{k+1}(\underline{x}) + L^{k+1}(\underline{y})$. $\square$

If $L(\underline{x}) = A\underline{x}$ and $L^{k+1}(\underline{x}) = L(L^k(\underline{x}))$, then

$L^n(\underline{x}) = L(L(\cdots(L(x)))) = AA\cdots A\underline{x} = A^n \underline{x}$.

Thus A^n is the matrix representing L^n.

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3. Find the kernel and range of the following linear operators on P_3.

1. $L(p(x)) = xp'(x)$,
2. $L(p(x)) = p(x) - p'(x)$,
3. $L(p(x)) = xp(0) + p(1)$.

1. $p(x) = a + bx + cx^2$

$L(p) = bx + 2cx^2$

Thus $\underline{\ker(L) = \text{span}\{1\}}$

$\underline{\text{im}(L) = \text{span}\{x, x^2\}}$.

2. $L(p) = a + bx + cx^2 - b - 2cx$

$= (a-b) + (b-2c)x + cx^2$

$\ker(L)$: $\left.\begin{array}{l} a-b = 0 \\ b-2c = 0 \\ c = 0 \end{array}\right\} \Rightarrow a = b = c = 0$ Thus $\underline{\ker(L) = \{0\}}$.

by the rank-nullity theorem, $\underline{\text{im}(L) = P_3}$.

3. $L(p) = ax + a + b + c$

$\ker(L)$: $\left.\begin{array}{l} a = 0 \\ a+b+c = 0 \end{array}\right\} \Rightarrow b = -c$

Thus $\underline{\ker(L) = \text{span}\{x^2 - x\}}$

$\underline{\text{im}(L) = \text{span}\{1, x\}}$

4. Let

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3.$$

Find the matrix A representing L with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$L(\underline{e}_1) = 1\,\underline{b}_1 + 0\,\underline{b}_2 + 1\,\underline{b}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_B$$

$$L(\underline{e}_2) = 0\,\underline{b}_1 + 1\,\underline{b}_2 + 1\,\underline{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_B$$

Thus, $\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}$

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5. Let $L:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix} \quad \text{and} \quad L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix},$$

find the value of $L((7,5)^T)$.

Let $\underline{u}_1 = \begin{pmatrix}1\\2\end{pmatrix}$ $\quad \underline{u}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$$U = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}, \quad U^{-1} = \frac{1}{-3}\begin{pmatrix}-1 & -1\\-2 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$$

$$\left[\begin{pmatrix}7\\5\end{pmatrix}\right]_{U} = U^{-1}\begin{pmatrix}7\\5\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}7\\5\end{pmatrix} = \frac{1}{3}\begin{pmatrix}12\\9\end{pmatrix}_U = \begin{pmatrix}4\\3\end{pmatrix}_U$$

Thus $L\begin{pmatrix}7\\5\end{pmatrix} = L(4\underline{u}_1 + 3\underline{u}_2) = 4L(\underline{u}_1) + 3L(\underline{u}_2)$

$$= 4\begin{pmatrix}-2\\3\end{pmatrix} + 3\begin{pmatrix}5\\2\end{pmatrix}$$

$$= \begin{pmatrix}-8\\12\end{pmatrix} + \begin{pmatrix}15\\6\end{pmatrix} = \begin{pmatrix}7\\18\end{pmatrix}.$$

6. For each $f \in C[0,1]$, define

$$L(f) = \int_0^t f(s)\,ds, \quad 0 \le t \le 1.$$

(a.) Show that L is a linear operator on $C[0,1]$.

(b.) Find the $L(e^t)$, $L(e^{-t})$, and $L(t^2)$.

(a.) $L(\alpha f + \beta g) = \int_0^t \alpha f(s) + \beta g(s)\,ds$

$= \int_0^t \alpha f(s)\,ds + \int_0^t \beta g(s)\,ds$

$= \alpha \int_0^t f(s)\,ds + \beta \int_0^t g(s)\,ds$

$= \alpha L(f) + \beta L(g).$ $\square$

(b) $L(e^t) = \int_0^t e^s\,ds = \boxed{e^t - 1}$

$L(e^{-t}) = \int_0^t e^{-s}\,ds = \boxed{-e^{-t} + 1}$

$L(t^2) = \int_0^t s^2\,ds = \boxed{\frac{1}{3}t^3}$

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7. Let L be the linear transformation mapping P_2 to $\mathbb{R}^2$ defined by

$$L(p) = \begin{pmatrix} \int_0^1 p(x)\,dx \\ p(0) \end{pmatrix}.$$

Find the matrix representing L with respect to the ordered bases $\{1, x\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$.

$$L(1) = \begin{pmatrix} \int_0^1 1\,dx \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L(x) = \begin{pmatrix} \int_0^1 x\,dx \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

Thus $\boxed{A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix}}$

8. Let S be the subspace of $C[a,b]$ spanned by e^x, xe^x, and x^2e^x. Let D be the differentiation operator on S.

(a.) Find the matrix representing D with respect to the ordered basis $\{e^x, xe^x, x^2e^x\} = \mathcal{U}$.

(b.) Find the inverse matrix $J = D^{-1}$.

(c.) Compute $\int x^2e^x\,dx$ and $J(x^2e^x)$. Compare the results.

(a.) $D(e^x) = e^x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{U}}$

$D(xe^x) = e^x + xe^x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{U}}$

$D(x^2e^x) = 2xe^x + x^2e^x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}_{\mathcal{U}}$

Thus $\boxed{D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}$

(b.) $D^{-1} = \boxed{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = J}$

(c.) $\int x^2e^x\,dx = \left\{ \begin{array}{cc} u & dv \\ +\ x^2 & e^x \\ -\ 2x & e^x \\ +\ 2 & e^x \\ -\ 0 & e^x \end{array} \right\} = x^2e^x - 2xe^x + 2e^x.$

$J(x^2e^x) = J\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}_{\mathcal{U}} = 2e^x - 2xe^x + x^2e^x.$

They are the same!

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9. Prove the following results.

1. If A is similar to B and B is similar to C, then A is similar to C.
2. If $A, B \in R^{n\times n}$ and $A \sim B$, then there exist matrices $S, T \in \mathbb{R}^{n\times n}$ with S nonsingular such that $A = ST$ and $B = TS$.
3. If $A \sim B$, then $\det A = \det B$.
4. If $A \sim B$, then $A^T \sim B^T$.
5. If $A \sim B$ and $\lambda \in \mathbb{R}$ is any number, then $(A - \lambda I) \sim (B - \lambda I)$.

1. $A \sim B \Rightarrow B = SAS^{-1}$ or $A = S^{-1}BS$

$B \sim C \Rightarrow B = T^{-1}CT$

Plugging in $A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS)$

$= (TS)^{-1} C (TS)$

Thus $A \sim C$. $\square$

2. $A \sim B \Rightarrow \exists\ S \in \mathbb{R}^{n\times n}$ nonsingular s.t. $SA = BS$.

Put $T = BS^{-1}$. Then $ST = SBS^{-1} = A$

and $TS = BS^{-1}S = B$. ✓ $\square$

3. $A \sim B \Rightarrow B = S^{-1}AS$

$\det(B) = \det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \frac{1}{\cancel{\det(S)}}\det(A)\cancel{\det(S)}$

$= \det(A)$. $\square$

4. $A \sim B \Rightarrow B = S^{-1}AS \Rightarrow B^T = (S^{-1}AS)^T = S^TA^T(S^{-1})^T = S^TA^T(S^T)^{-1}$. $\square$

Thus $B^T \sim A^T$.

5. $A \sim B \Rightarrow B = S^{-1}AS$

$S^{-1}(A - \lambda I)S = S^{-1}AS - \lambda S^{-1}IS = B - \lambda I$. $\square$

10. Let L be the linear operator on $\mathbb{R}^3$ given by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix},$$

and let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

Find the matrix B representing L with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$L(v_1) = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_V$$

$$L(v_2) = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_V$$

$$L(v_3) = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_V$$

so $\boxed{B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$