Math 511: Linear Algebra
Good Problems 4

Due: Friday, 27 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Consider the vectors

$$\mathbf{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix}-1\\1\end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}2\\5\end{pmatrix}.$$

(*a*.) Find the transition matrix from the ordered basis $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ to the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2\}$.

From U to E:

$$\boxed{U = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}}$$

(*b*.) Find the transition matrix from the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ to the ordered basis $V = \{\mathbf{v}_1, \mathbf{v}_2\}$.

From E to V:

$$V = \begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}$$

$$V^{-1} = \frac{1}{5-4}\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}, \quad \boxed{V^{-1} = \begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}}$$

(*c*.) Find the transition matrix from the ordered basis $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ to the ordered basis $V = \{\mathbf{v}_1, \mathbf{v}_2\}$.

From U to V: $S = V^{-1}U$.

$$S = V^{-1}U = \begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} =$$

$$\boxed{S = \begin{pmatrix}3 & -7\\-1 & 3\end{pmatrix}}$$

2. (*a.*) Use the map $\Gamma : P_2 \to \mathbb{R}^2$ to show that $E = \{1, x\}$ is the standard basis for P_2; *i.e.*, it corresponds to the standard basis of $\mathbb{R}^2$.

$\Gamma(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \Gamma(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\Gamma(1) = \underline{e}_1$ and $\Gamma(x) = \underline{e}_2$

Thus $\{1, x\}$ is the standard basis of P_2.

(*b.*) Show that $V = \{2x - 1, 2x + 1\}$ forms another basis for P_2.

$\Gamma(2x-1) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad \Gamma(2x+1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$V = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$; $\det(V) = -2 - 2 = -4 \neq 0$

Thus V forms a basis for P_2.

(*c.*) Find the transition matrix from E to V.

$$V^{-1} = \frac{1}{-4}\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = V^{-1}}$$

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3. (*a.*) Find the transition matrix representing the change of coordinates in P_3 from the standard basis $E = \{1, x, x^2\}$ to the ordered basis $V = \{1, 1-x, 1+x-x^2\}$.

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{V^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}$$

by inspection. You can also use RREF methods, or cofactors.

(*b.*) Write the vector $p(x) = 3 + 2(1-x) - 3(1+x-x^2)$ in standard coordinates by using the matrix in part (*a.*). Check your answer by distributing and combining like terms.

$$[p]_V = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}_V$$

$$p = V[p]_V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}_V = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

so $\boxed{p(x) = 2 - 5x + 3x^2}$

4. Consider the vectors in $\mathbb{R}^3$.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ -8 \\ -5 \end{pmatrix}.$$

(a.) Show that $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ forms a basis for $\mathbb{R}^3$.

$$X = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\det(X) = 1\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} = 1(-3) + 2(-1+3) = -3+4 = 1 \neq 0.$$

So $\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ form a basis for $\mathbb{R}^3$.

(b.) Find the transition matrix from $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ to the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -6 \\ 0 & 3 & 0 & -3 & 3 & -6 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -6 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right)$$

$$\boxed{X = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ -1 & 1 & -1 \end{pmatrix}}$$

For Part c.), need:

$$X^{-1} = \begin{pmatrix} -3 & 2 & -6 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}$$

(c.) Write $\mathbf{v}$ as a linear combination of $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$.

$$[V]_X = X^{-1}v = \begin{pmatrix} -3 & 2 & -6 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -8 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

So $\boxed{\underline{v} = 2\underline{x}_1 - 2\underline{x}_2 + \underline{x}_3}$

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5. Consider the matrix

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}.$$

Find bases for the row, column, and null spaces of A.

$$\begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \xrightarrow{R_1 - R_3 \to R_3} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 0 & -7 & -1 & 2 \end{pmatrix} \xrightarrow{R_1 + 3R_2 \to R_2} \begin{pmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & -7 & -1 & 2 \end{pmatrix}$$

$R_2 + R_3 \to R_3$

$$\to \begin{pmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} -3 & 1 & 0 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -21 & 0 & 0 & 26 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$-R_3 \to R_3$

$R_1 - 3R_3 \to R_1$

$7R_1 - R_2 \to R_1$

$$\to \begin{pmatrix} 1 & 0 & 0 & \frac{-26}{21} \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

For $N(A)$:

$$\begin{aligned} x_1 &= 26/21\,\alpha \\ x_2 &= 2/7\,\alpha \\ x_3 &= 0 \\ x_4 &= \alpha \end{aligned} \quad \to \quad \begin{matrix} 26\alpha \\ 6\alpha \\ 0 \\ 21\alpha \end{matrix}$$

$$\boxed{\begin{aligned} \text{Row}(A) &= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{-26}{21} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \\ \text{Col}(A) &= \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\} = \mathbb{R}^3. \\ N(A) &= \text{span}\left\{ \begin{pmatrix} 26 \\ 6 \\ 0 \\ 21 \end{pmatrix} \right\}. \end{aligned}}$$

6. Determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the following vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & 4 & 5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\text{Rank}(X) = 3$,

so $\dim\left(\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}\right) = 3$.

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7. Recall that a matrix equation $A\mathbf{x} = \mathbf{b}$ is said to be *consistent* if it has at least one solution, and *inconsistent* if it has no solutions.

(*a.*) With our new perspective, a matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A. Prove this claim.

$A = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$

$A\underline{x} = x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n$

Then $A\underline{x} = \underline{b}$ if and only if $x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n = \underline{b}$, whence $\underline{b}$ is a linear combination of the columns of A. Equivalently, $\underline{b} \in Col(A)$. $\square$

(*b.*) Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of $(A|\mathbf{b})$ is equal to the rank of A.

By the previous problem, $A\underline{x} = \underline{b}$ is consistent iff $\underline{b}$ is a linear combination of the columns of A.

Thus $(A|\underline{b})$ and (A) will have the same number of linearly independent columns, hence the same rank.

8. Let $A \in \mathbb{R}^{m\times n}$ with $m < n$ and $\text{rank}(A) = m$, and let $\mathbf{b}$ be any vector in $\mathbb{R}^m$. (*a.*) Explain why the system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

If $\text{rank}(A) = m$, then $Col(A) = \mathbb{R}^m$.

By the previous problems, $\underline{b} \in \mathbb{R}^m = Col(A)$

$\Rightarrow$ $A\underline{x} = \underline{b}$ is consistent.

Now, since $m < n$, then there will always be free variables,

so the solutions of $A\underline{x} = \underline{b}$ will span an $(n-m) > 0$ dimensional subspace of $\mathbb{R}^n$.

(*b.*) What can you say about the nature of the solutions if $\text{rank}(A) < m$?

If $\text{rank}(A) < m$, then $A\underline{x} = \underline{b}$ could have either ∞-many solutions or zero solutions, depending on whether $\underline{b} \in Col(A)$ or not.

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9. Let $\mathbf{a} \in \mathbb{R}^2$ be a fixed non-zero vector. A mapping of the form

$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$

is called a *translation*. (*a*.) Show that a translation is *not* a linear operator.

L1. $L(\alpha \underline{x}) = (\alpha \underline{x}) + \underline{a} = \alpha \underline{x} + \underline{a} \; \boxed{\neq} \; \alpha(\underline{x} + \underline{a}) = \alpha L(\underline{x})$.

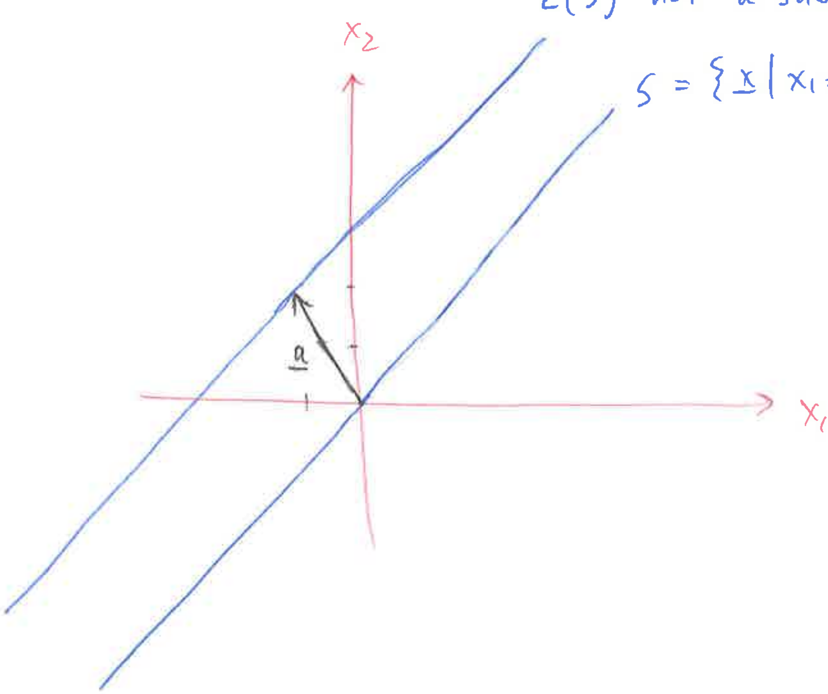
L2. $L(\underline{x} + \underline{y}) = (\underline{x} + \underline{y}) + \underline{a} = \underline{x} + \underline{y} + \underline{a} \neq (\underline{x} + \underline{a}) + (\underline{y} + \underline{a}) = L(\underline{x}) + L(\underline{y})$

So both L1 and L2 fail.

(*b*.) Let $\mathbf{a} = (-1, 2)^T$. Illustrate geometrically the effect of the translation $L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ on the subspace $S = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 = x_1\}$.

$L(S)$ not a subspace!

$S = \{\underline{x} \mid x_1 = x_2\}$.



10. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$. Justify your answers (Prove or provide a counterexample).

1. $L(\mathbf{x}) = (x_2, x_3)^T$;
2. $L(\mathbf{x}) = \mathbf{0}$;
3. $L(\mathbf{x}) = (1 + x_1, x_2)^T$;
4. $L(\mathbf{x}) = (x_1, x_2 + x_3)^T$;
5. $L(\mathbf{x}) = (x_1, 0)^T$;
6. $L(\mathbf{x}) = (x_1, 1)^T$.

1. Yes - verify.
2. Yes - verify
3. No.

$$L(\alpha \mathbf{x}) = L\begin{pmatrix}\alpha x_1\\ \alpha x_2\\ \alpha x_3\end{pmatrix} = \begin{pmatrix}1+\alpha x_1\\ \alpha x_2\end{pmatrix} = \alpha\begin{pmatrix}1/\alpha + x_1\\ x_2\end{pmatrix} \neq \alpha L(\mathbf{x}) \,!$$

similarly $L(\mathbf{x}+\mathbf{y}) \neq L(\mathbf{x}) + L(\mathbf{y})$.

4. Yes. - verify.
5. Yes - verify.
6. No.

$$L(\mathbf{x}+\mathbf{y}) = L\begin{pmatrix}x_1+y_1\\ x_2+y_2\\ x_3+y_3\end{pmatrix} = \begin{pmatrix}x_1+y_1\\ 1\end{pmatrix} = \begin{pmatrix}x_1\\ 1/2\end{pmatrix} + \begin{pmatrix}y_1\\ 1/2\end{pmatrix} \neq \begin{pmatrix}x_1\\ 1\end{pmatrix} + \begin{pmatrix}y_1\\ 1\end{pmatrix} = L(\mathbf{x}) + L(\mathbf{y})$$

and similarly $L(\alpha\mathbf{x}) \neq \alpha L(\mathbf{x})$.