Math 511: Linear Algebra
Good Problems 4.5

Due: Friday, 27 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & -1 & 4 & 6 \\ 2 & 6 & 4 & 1 & 5 & 12 \\ -1 & -3 & 0 & 1 & -2 & -4 \end{pmatrix}.$$

Find bases for the row, column, and null spaces of A.

RREF:

$$\rightarrow \begin{pmatrix} 1 & 3 & 2 & -1 & 4 & 6 \\ 0 & 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 2 & 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 & -1 & 4 & 6 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

$$Row(A) = span\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

$$Col(A) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^3.$$

$$N(A): \quad span\left\{ \begin{pmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$x_6 = \alpha$

$x_5 = \beta$

$x_4 = x_5 = \beta$

$x_3 = -\alpha - \beta$

$x_2 = \gamma$

$x_1 = -3\gamma - \beta - 4\alpha$

2. Consider the subset of $C(-\infty,\infty)$ defined by $S = \{f \in C(-\infty,\infty) \mid f(0) = f(5) = 0\}$. Prove that S is a subspace of $C(-\infty,\infty)$.

Let $f, g \in S$ and $\alpha \in \mathbb{R}$.

S1. $(\alpha f)(0) = \alpha \cdot f(0) = \alpha \cdot 0 = 0$

$(\alpha f)(5) = \alpha \cdot f(5) = \alpha \cdot 0 = 0$

thus $\alpha f \in S$.

S2. $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$

$(f+g)(5) = f(5) + g(5) = 0 + 0 = 0$.

Thus $f+g \in S$. $\square$

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3. Are the vectors $e^x \sin(x)$ and $e^x \cos(x)$ linearly independent or linearly dependent in $C(-\pi, \pi)$? Justify your answer.

Take the Wronskian:

$$W(e^x \sin x, e^x \cos x) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & -e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x}\left[-\sin^2 x + \cancel{\sin x \cos x} - \cancel{\sin x \cos x} - \cos^2 x\right] = e^{2x}[-1] = -e^{2x}$$

$$\neq 0.$$

Thus $e^x \sin x$ and $e^x \cos x$ are linearly ind.

4. Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. (*a.*) Verify that $\text{span}\{e^x, e^{-x}\} = \text{span}\{\cosh(x), \sinh(x)\}$. (*b.*) Find the transition matrix from $E = \{e^x, e^{-x}\}$ to $U = \{\cosh(x), \sinh(x)\}$. (*c.*) Write e^x and e^{-x} as linear combinations of $\cosh(x)$ and $\sinh(x)$.

a.) In E coord's $e^x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e^{-x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$\cosh x = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $\sinh x = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$.

$$\det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2} \neq 0.$$

Thus ~~these~~ U forms a basis, hence spans $\{e^x, e^{-x}\}$.

b.) From E to U $= U^{-1}$

$$U^{-1} = -2 \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

c.) $[e^x]_U = U^{-1} e^x = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_U = \cosh x + \sinh x$

$[e^{-x}]_U = U^{-1} e^{-x} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_U = \cosh x - \sinh x$.

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5. Find all values of α for which the linear system is inconsistent.

$$\begin{pmatrix} 1 & \alpha & 2 \\ -2 & 1 & 4\alpha - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\left(\begin{array}{ccc|c} 1 & \alpha & 2 & 3 \\ -2 & 1 & 4\alpha-2 & 2 \end{array}\right)$ is inconsistent iff $c \cdot R_1 = R_2$.

$-2R_1 \to R_1 \longrightarrow \left(\begin{array}{ccc|c} -2 & -2\alpha & -4 & -6 \\ -2 & 1 & 4\alpha-2 & 2 \end{array}\right)$ $\quad -6 \neq 2.$

$\left.\begin{array}{l} -2\alpha = 1 \\ 4\alpha - 2 = -4 \end{array}\right\}$ $\boxed{\alpha = -\tfrac{1}{2}}$

6. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/4 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/4 \end{array}\right)$$

$$\to \boxed{A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 0 & -1/2 \\ 0 & 0 & -1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 & 1/4 \end{pmatrix}}$$

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7. Consider the polynomial $p(x) = x^3 - x + 1$ in P_4. Write p as a linear combination of $(x+1)^3$, $(x+1)^2$, $(x+1)$, and 1. Explain what you just did in the language of calculus.

$p = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ in std coords.

$(x+1)^3 = 1 + 3x + 3x^2 + x^3 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$

$(x+1)^2 = 1 + 2x + x^2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

$(x+1) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad U^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[p]_u = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}_u = \boxed{-1 + 2(x+1) - 3(x+1)^2 + (x+1)^3}$

This is the Taylor Series for $p \in P_4$ centered at $x_0 = -1$!

8. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be an *involution* if $A^2 = I$.

Show that the following matrix is an involution for any value of θ.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & -\cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$A^2 = I$ straight forward.

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9. Find the determinant of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -3 \\ 0 & 7 & 17 \\ 0 & 0 & 2 \end{pmatrix}.$$

$\det(A) = 1 \cdot 42 = 42.$

10. (*a.*) Prove that any finite set of vectors that contains the zero vector is linearly dependent.

If $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n, \underline{0}\}$

Then $0 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2 + \dots + 0 \cdot \underline{v}_n + 1 \cdot \underline{0}_v = \underline{0}$

but $c_{n+1} = 1 \neq 0$.

Thus the vectors in S must be linearly dep. $\square$

(*b.*) Suppose the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ forms a basis for a vector space V. If $\mathbf{v} \in V$ is any other vector, prove that the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{v}\}$ is linearly dependent.

Since $X = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ is a basis, then

$\underline{v} = c_1 \underline{x}_1 + c_2 \underline{x}_2 + \dots + c_n \underline{x}_n$

subtracting $\underline{v}$ yields:

$c_1 \underline{x}_1 + c_2 \underline{x}_2 + \dots + c_n \underline{x}_n - 1 \cdot \underline{v} = \underline{0}$

so $c_{n+1} = -1 \neq 0$ and

the vectors are linearly dep.