Math 511: Linear Algebra
Good Problems 3

Due: Friday, 20 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name:______________________________

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. (*a.*) Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 3 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\det(BC) = \det(B)\det(C)$

so $\det A = 1 \cdot 6 = \boxed{6}$

(*b.*) Find the determinant of the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 4 & 0 & 0 & -7 \\ -3 & 0 & 1 & 2 \\ -5 & 0 & 0 & 11 \end{pmatrix}.$$

$\det(B) = (-1) \cdot 1 \cdot \begin{vmatrix} 4 & 0 & -7 \\ -3 & 1 & 2 \\ -5 & 0 & 11 \end{vmatrix}$

$= -1 \cdot 1 \cdot \begin{vmatrix} 4 & -7 \\ -5 & 11 \end{vmatrix}$

$= -1(44 - 35) = \boxed{-9}$

2. (*a*.) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

$AA^{-1} = I$ so

$1 = \det(I) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$

w/ $\det(A) \neq 0$.

so dividing by $\det(A)$ yields

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

(*b*.) Let $A, B \in \mathbb{R}^{n\times n}$ be nonsingular. Show that AB is also nonsingular.

one way: $\det(A) \neq 0$, $\det(B) \neq 0$

so $0 \neq \det(A)\det(B) = \det(AB) \Rightarrow AB$ nonsingular.

Another way:

$(B^{-1}A^{-1})\cdot(AB) = B^{-1}(A^{-1}A)B = B^{-1}(IB) = B^{-1}B = I$

and $(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(IA^{-1}) = AA^{-1} = I$

so $(AB)^{-1} = B^{-1}A^{-1}$ exists.

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3. Consider the set of vectors in $\mathbb{R}^3$ given by

$$S = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

(a.) Show that S forms a basis for $\mathbb{R}^3$.

$$S = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\det(S) = -1\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = -1(2) + 1(1) = 1-2 = -1 \neq 0.$$

Thus the columns of S span $\mathbb{R}^3$ and are linearly independent.

(b.) Write the vector $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ as a linear combination of the basis vectors in S.

$$\underline{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & +1 & -1 & -1 \\ 0 & +1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$-1R_1 \to R_1$
$-(R_1+R_3) \to R_3$
$-1R_2 \to R_2$

$R_2 + R_3 \to R_3$
$R_2 \leftrightarrow$ new R_3

$R_1 + R_3 \to R_1$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

so $c_1 = -1$, $c_2 = 0$, $c_3 = 1$

and

$$\boxed{\underline{b} = (-1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + 1\cdot\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}$$

4. Let $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$. $\leftarrow$ $c_1\underline{x}_1 + c_2\underline{x}_2 + c_3\underline{x}_3 = \underline{0}$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$.

(*a.*) Let

$$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{x}_1 + \mathbf{x}_3.$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Justify your answer.

Assume $a_1\underline{y}_1 + a_2\underline{y}_2 + a_3\underline{y}_3 = \underline{0}$.

then $a_1(\underline{x}_1 + \underline{x}_2) + a_2(\underline{x}_2 + \underline{x}_3) + a_3(\underline{x}_1 + \underline{x}_3) = \underline{0}$

or $(a_1 + a_3)\underline{x}_1 + (a_1 + a_2)\underline{x}_2 + (a_2 + a_3)\underline{x}_3 = \underline{0}$

$$\Rightarrow \left.\begin{array}{l} a_1 + a_3 = 0 \\ a_1 + a_2 = 0 \\ a_2 + a_3 = 0 \end{array}\right\}$$

Solving this system yields $a_1 = a_2 = a_3 = 0$.

Thus $\underline{y}_1, \underline{y}_2, \underline{y}_3$ are linearly independent

(*b.*) Let

$$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \quad \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1.$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Justify your answer.

Notice that $\underline{y}_1 + \underline{y}_2 = (\underline{x}_2 - \underline{x}_1) + (\underline{x}_3 - \underline{x}_2)$

$= \underline{x}_3 - \underline{x}_1$

$= \underline{y}_3$.

Thus $1 \cdot \underline{y}_1 + 1 \cdot \underline{y}_2 + (-1)\,\underline{y}_3 = \underline{0}$.

Therefore $\underline{y}_1, \underline{y}_2, \underline{y}_3$ are linearly dependent.

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5. Determine whether the vectors $\{x+2, x^2-1\}$ are linearly independent in P_3. Do they form a basis for P_3? Justify your answer.

$$c_1(x+2) + c_2(x^2-1) = 0x^2 + 0x + 0$$

$$c_2 x^2 + c_1 x + (2c_1 - c_2) = 0$$

$$\left.\begin{array}{l} c_2 = 0 \\ c_1 = 0 \end{array}\right\} \text{ so } x+2, x^2-1 \text{ are } \underline{\text{linearly}}\ \underline{\text{independent}}.$$

They cannot span P_3, so they do not form a basis.

6. Show that the vectors $\{\cos(x), 1, \sin^2(x/2)\}$ are linearly dependent in $C[-\pi,\pi]$.

Trig identity:

$$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos x$$

So $\frac{1}{2}\cdot 1 + \left(\frac{-1}{2}\right)\cdot \cos x + (-1)\cdot \sin^2\left(\frac{x}{2}\right) = 0$ for all x.

Therefore $\{1, \cos x, \sin^2\left(\frac{x}{2}\right)\}$ are <u>linearly</u> <u>dependent</u> in $C[-\pi,\pi]$.

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7. Consider the vectors $f(x) = x^2$ and $g(x) = x|x|$.

(a.) Show that f and g are linearly independent in $C(-\infty,\infty)$.

Write $g(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

$x \geq 0$: $\quad c_1 f(x) + c_2 g(x) = c_1 x^2 + c_2 x^2 = 0$

$x < 0$: $\quad c_1 f(x) + c_2 g(x) = c_1 x^2 - c_2 x^2 = 0$

$\Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{matrix} \Rightarrow c_1 = c_2 = 0.$

So they are linearly independent.

(b.) Show that f and g are linearly dependent in $C(-\infty, 0)$.

In this domain $g(x) = -x^2 = -f(x)$.

Thus $(+1) f(x) + (1) g(x) = 0$.

and f, g are linearly dependent in $C(-\infty, 0)$.

8. Let $A \in \mathbb{R}^{m\times n}$. Show that if A has linearly independent column vectors, then $N(A) = \{\mathbf{0}\}$.

Let $A = \left(\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n\right)$

where $\underline{a}_i \in \mathbb{R}^m$.

The $\underline{a}_i$'s are linearly independent implies

$$c_1\underline{a}_1 + c_2\underline{a}_2 + \cdots + c_n\underline{a}_n = \underline{0} \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0.$$

That is, $A\underline{c} = \underline{0}$, $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

has only one solution, $\underline{c} = \underline{0}$.

The set of all $\underline{c} \in \mathbb{R}^n$ that solve $A\underline{c} = \underline{0}$ is the null space of A.

Thus $N(A) = \{\underline{0}\}$.

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9. In $C[-\pi,\pi]$, find the dimension of the subspace spanned by the vectors 1, $\cos(2x)$, and $\cos^2(x)$. Justify your answer.

Comparing w/ problem 6.,

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

So these vectors are linearly dependent, hence do not span a 3-dim subspace.

But $W[1, \cos(2x)] = \begin{vmatrix} 1 & \cos(2x) \\ 0 & -2\sin(2x) \end{vmatrix} = -2\sin(2x) \neq 0$ if $x = \pi/4$ (for example).

Therefore 1, $\cos(2x)$ are linearly independent, and thus span a 2-dimensional subspace of $C[-\pi,\pi]$.

10. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}.$$

(*a.*) Show that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent. (*b.*) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. (*c.*) What is the dimension of span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? (*d.*) Give a geometric description of span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

a) Put $X = \begin{pmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{pmatrix}$.

$$\det(X) = 2(-4-24) - 3(4-18) + 2(4+3)$$

$$= -56 + 42 + 14 = 0.$$

Thus x_1, x_2, x_3 are linearly dependent.

b.) $c_1\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2c_1 + 3c_2 \\ c_1 - c_2 \\ 3c_1 + 4c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & -1 & 0 \\ 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad c_1 = c_2 = 0.$

So x_1 and x_2 are linearly independent.

c.) $\dim(\text{span}\{x_1, x_2, x_3\}) = 2.$

d.) span$\{x_1, x_2, x_3\}$ is a plane through the origin in $\mathbb{R}^3$.