Math 511: Linear Algebra
Good Problems 2

Due: Friday, 13 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: Key

Instructions: Complete all 10 problems. Each problem is worth 10 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

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1. Consider a linear system whose matrix equation is of the form

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

For what values of α will the system have a unique solution?

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & \alpha \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The system $A\underline{x} = \underline{b}$ has a unique sol'n iff $\det(A) \neq 0$.

$$\det(A) = 1 \cdot \begin{vmatrix} 4 & 3 \\ -2 & \alpha \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 2 & \alpha \end{vmatrix} + 1 \begin{vmatrix} -1 & 4 \\ 2 & -2 \end{vmatrix}$$

$$= 1(4\alpha + 6) - 2(-\alpha - 6) + 1(2 - 8)$$

$$= 4\alpha + \cancel{6} + 2\alpha + 12 - \cancel{6}$$

$$= 6\alpha + 12$$

$$= 0 \text{ iff } \alpha = -2.$$

Therefore the system has a unique solution for all values of α except -2, or

$$\boxed{\alpha \neq -2.}$$

2. Given the linear systems

$$(a)\ \begin{cases} x_1 + 2x_2 = 2, \\ 3x_1 + 7x_2 = 8, \end{cases} \quad \text{and} \quad (b)\begin{cases} x_1 + 2x_2 = 1, \\ 3x_1 + 7x_2 = 7. \end{cases}$$

Solve both systems simultaneously by incorporating the right-hand sides into a 2×2 matrix B and computing the reduced row echelon form of

$$(A \mid B) = \left(\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right).$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right) \underset{3R_1 - R_2 \to R_2}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 0 & -1 & -2 & -4 \end{array}\right) \underset{-R_2 \to R_2}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right) \underset{R_1 - 2R_2 \to R_1}{\longrightarrow}$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

Thus the sol'n to system (a) is $\underline{x} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

and the sol'n of system (b) is $\underline{x} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$.

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3. (*a.*) Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Show that if $d = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

$$A^{-1} = \frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

$$A^{-1}A = \frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{d}\begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

similary, $AA^{-1} = \frac{1}{d}\begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$ □

(*b.*) Use this formula to compute the inverse of each of the following matrices:

$$A = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7-6}\begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}$$

$$B^{-1} = \frac{1}{9-10}\begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$$

$$C^{-1} = \frac{1}{8-6}\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3/2 \\ -1 & 2 \end{pmatrix}$$

4. Let

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Compute A^2 and A^3. What will A^n turn out to be?

$$A^2 = AA = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = A.$$

$A^3 = A(AA) = AA^2 = AA = A^2 = A$ by the previous computation.

(You should compute it explicitly.)

$A^n = A$ by the previous argument.

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5. A matrix is said to be *idempotent* if $A^2 = A$.

(a.) Show that if A is idempotent, then $I - A$ is also idempotent.

$$(I-A)^2 = (I-A)(I-A)$$
$$= I - IA - AI + A^2$$
$$= I - A - A + A$$
$$= I - A. \quad \square$$

(b.) Show that if A is idempotent, then $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$.

$$(I+A)\cdot(I - \tfrac{1}{2}A) = I - \tfrac{1}{2}A + A - \tfrac{1}{2}A^2$$
$$= I - \tfrac{1}{2}A + A - \tfrac{1}{2}A$$
$$= I - A + A$$
$$= I.$$

$(I - \frac{1}{2}A)(I + A) = \ldots$ similar calculation $= I. \quad \square$

6. Let $A \in \mathbb{R}^{n\times n}$, and let $B = A + A^T$ and $C = A - A^T$.

$(a.)$ Show that B is symmetric and C is skew symmetric.

$$(B)_{ij} = (A+A^T)_{ij} = (A)_{ij} + (A^T)_{ij} = a_{ij} + a_{ji}$$

$$(B)_{ji} = (A+A^T)_{ji} = (A)_{ji} + (A^T)_{ji} = a_{ji} + a_{ij} = a_{ij} + a_{ji} = (B)_{ij}.$$

So B is symmetric.

$$(C)_{ij} = (A-A^T)_{ij} = (A)_{ij} - (A^T)_{ij} = a_{ij} - a_{ji} =$$

$$= -(a_{ji} - a_{ij}) = -\left((A)_{ji} - (A^T)_{ji}\right) = -(C)_{ji}$$

So C is skew-symmetric. $\square$

$(b.)$ Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

Notice that $B+C = A + A^T + A - A^T = 2A.$

Thus $A = \frac{1}{2}(B+C) = \frac{1}{2}B + \frac{1}{2}C = \underbrace{\frac{1}{2}(A+A^T)}_{\text{symm}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{skew symm}}$

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7. Determine the null space of the matrix.

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$R_1 + R_3 \to R_3$

$2R_1 - R_2 \to R_2$

$R_2 + R_3 \to R_3$

$R_1 + R_2 \to R_1$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now x_4, x_2 are free.

Put

$$x_1 = -\beta - 5\alpha$$
$$x_2 = \beta$$
$$x_3 = -3\alpha$$
$$x_4 = \alpha$$

So the null space is all $\underline{x} \in \mathbb{R}^4$ that look like

$$\underline{x} = \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

or

$$\boxed{N(A) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}}$$ in our new language.

8. Determine whether the following are subspaces of P_4 (Be careful!):

1. The set of polynomials in P_4 of even degree;
2. The set of all polynomials of degree 3;
3. The set of all polynomials $p \in P_4$ such that $p(0) = 0$;
4. The set of all polynomials in P_4 having at least one real root.

If the answer is "yes," you must verify the axioms **S1** and **S2**. If the answer is "no," you must come up with an example that shows that (at least) one of the two axioms fails.

1.) No!

Counter example: let $p(x) = x^2 + x$, $q(x) = x^2$

Then $p, q \in S$, but $(p-q)(x) = -x \notin S$.

2.) No!

Counter example: $p(x) = x^3 + x^2$, $q(x) = x^3$

Then $p, q \in S$ but $(p-q)(x) = -x^2 \notin S$.

3.) Yes!

Proof. Let $p, q \in S$, $\alpha \in \mathbb{R}$.

S1. $(\alpha p)(0) = \alpha (p(0)) = \alpha \cdot 0 = 0$. So $(\alpha p) \in S$.

S2. $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$ so $(p+q) \in S$. $\square$

4.) No!

Counter example.

let $p(x) = x^2$ and $q(x) = x + 1$

roots $(p) = 0$, roots $(q) = -1$, so $p, q \in S$.

but $(p+q)(x) = x^2 + x + 1$, roots $(p+q) = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \notin \mathbb{R}$!

so $(p+q) \notin S$.

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9. Which of the sets are spanning sets for $\mathbb{R}^3$? Justify your answers.

1. $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$;
2. $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$;
3. $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$;
4. $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$;
5. $\{(1,1,3)^T, (0,2,1)^T\}$.

1.) Yes.

$\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0.$

2.) Yes.

Since 1.) spans, so does 2.).

3.) No.

$\det\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{pmatrix} = 2\begin{vmatrix} 2 & 2 \\ -2 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = 2(4) - 3(4) + 2(2) = 8 - 12 + 4 = 0.$

4.) No.

~~det~~ $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = -\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ There are only 3 vectors and one is a multiple of another, so they cannot span $\mathbb{R}^3$.

5.) No.

There are only 2 vectors!

10. Determine whether the following sets of vectors are linearly independent in $\mathbb{R}^3$:

1. $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$;
2. $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$;
3. $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$;
4. $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$;
5. $\{(1,1,3)^T, (0,2,1)^T\}$.

1. Yes.

$\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0.$

2. No.

Four vectors in $\mathbb{R}^3$ cannot be linearly independent.

3. No.

det = 0. (see last problem.)

4. No.

$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = -\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ so $1\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + 1\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} = \underline{0}.$

5. Yes

$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \cancel{3} & \cancel{1} & \cancel{0} \end{array}\right)$ $\boxed{\begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}}$