Math 511: Linear Algebra
Good Problems 1

Due: Friday, 6 June 2014

LATE SUBMISSIONS WILL NOT BE ACCEPTED

Name: KEY

Instructions: Complete all 8 problems. Each problem is worth 12 points.

Show *enough* work on the paper provided (this paper), and follow all instructions carefully. Write your name on each page.

You may use any electronic (or other) aids that you wish, but you are expected to show all relevant details of any calculations. A correct "answer" is not good enough; I need to see how you got it!

Good Luck!

Name:______________________________

1. Let P_n be the set of all polynomials of degree less than or equal to $n-1$. Verify axioms **C1**, **C2**, **A3**, **A4**, and **A6** of the definition of a vector space.

C1. Let $p \in P_n$, $\alpha \in \mathbb{R}$. $p(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$

$(\alpha p)(x) = \alpha a_1 + \alpha a_2 x + \alpha a_3 x^2 + \cdots + \alpha a_n x^{n-1}$

$\alpha a_i \in \mathbb{R}$ whenever $a_i \in \mathbb{R}$, so $\alpha p \in P_n$. ✓

C2. Let $p, q \in P_n$

$$\begin{aligned} & p(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} \\ + \; & q(x) = b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1} \\ \hline & (p+q)(x) = (a_1+b_1) + (a_2+b_2)x + (a_3+b_3)x^2 + \cdots + (a_n+b_n)x^{n-1} \end{aligned}$$

$a_i + b_i \in \mathbb{R}$ whenever $a_i, b_i \in \mathbb{R}$, so $(p+q) \in P_n$.

A3. $\underline{0} \in P_n$ is given by $\underline{0}(x) = 0 + 0x + 0x^2 + \cdots + 0x^{n-1}$

Indeed $(p + \underline{0})(x) = (a_1 + 0) + (a_2 + 0)x + (a_3 + 0)x^2 + \cdots + (a_n + 0)x^{n-1}$

$= a_1 + a_2 x + \cdots + a_n x^{n-1}$

$= p(x)$.

A4. Let $p \in P_n$ s.t. $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$

$-p \in P_n$ is given by $(-p)(x) = -a_1 + (-a_2)x + \cdots + (-a_n)x^{n-1}$

Indeed $p + (-p) = \underline{0}$ in this case.

A6. let $\alpha, \beta \in \mathbb{R}$, $p \in P_n$. $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$

$$\begin{aligned} & (\alpha p)(x) = \alpha a_1 + \alpha a_2 x + \cdots + \alpha a_n x^{n-1} \\ + \; & (\beta p)(x) = \beta a_1 + \beta a_2 x + \cdots + \beta a_n x^{n-1} \\ \hline & (\alpha p + \beta p)(x) = (\alpha+\beta)a_1 + (\alpha+\beta)a_2 x + \cdots + (\alpha+\beta)a_n x^{n-1} = (\alpha+\beta)p(x) \end{aligned}$$

Thus, $\alpha p + \beta p = (\alpha+\beta)p$.

2. Let $\Gamma : P_n \to \mathbb{R}^n$ be the map defined by $\Gamma(p) = (a_1, a_2, \ldots, a_n)^T$, where $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$. Verify that $\Gamma(p+q) = \Gamma(p) + \Gamma(q)$ and $\Gamma(\alpha p) = \alpha\Gamma(p)$ for all $p, q \in P_n$ and all $\alpha \in \mathbb{R}$.

Let $\alpha, \beta \in \mathbb{R}$.

$p, q \in P_n$ s.t. $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$

$q(x) = b_1 + b_2 x + \cdots + b_n x^{n-1}$

$\Gamma(p) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\Gamma(q) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, $\Gamma(p) + \Gamma(q) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$

$(p+q)(x) = (a_1 + b_1) + (a_2 + b_2)x + \cdots + (a_n + b_n) x^{n-1}$

$\Gamma(p+q) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} = \Gamma(p) + \Gamma(q)$

so $\Gamma(p+q) = \Gamma(p) + \Gamma(q)$ □

Meanwhile,

$(\alpha p)(x) = (\alpha a_1) + (\alpha a_2)x + \cdots + (\alpha a_n) x^{n-1}$

so $\Gamma(\alpha p) = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \alpha \Gamma(p)$.

Name:______________________________

3. Let $\mathbb{Z}$ denote the set of all integers with addition defined in the usual way, and scalar multiplication defined by

$$\alpha \circ k = [\![\alpha]\!] \cdot k$$

for all $\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$, where $[\![\alpha]\!]$ denotes the greatest integer less than or equal to α. Show that $\mathbb{Z}$ together with these operations is *not* a vector space.

List all axioms which fail to hold, and give an example of each.

C1, C2, A1, A2, A3, A4, A5, and A8 are all ok.

A6 and A7 fail.

<u>A6:</u> Let $\alpha = \beta = 1.5$, and let $k \in \mathbb{Z}$.

Then $\alpha + \beta = 3$.

Now, $\alpha \circ k = [\![1.5]\!] k = k$

$+ \ \beta \circ k = [\![1.5]\!] k = k$

$\alpha \circ k + \beta \circ k = 2k$

but $(\alpha+\beta) \circ k = [\![3]\!] k = 3k$.

$3k \neq 2k$, so A6 fails.

<u>A7:</u> Let $\alpha = \frac{4}{3}$ and $\beta = 3$, so that $\alpha\beta = \frac{4}{3} \cdot 3 = 4$.

Then $(\alpha\beta) \circ k = [\![4]\!] k = 4k$

but $\alpha \circ (\beta \circ k) = [\![\frac{4}{3}]\!] \cdot [\![3]\!] k = 1 \cdot 3 \cdot k = 3k$.

$3k \neq 4k$, so A7 fails.

4. Let $A \in \mathbb{R}^{m \times n}$. Explain why the matrix multiplications $A^T A$ and AA^T are always possible.

Does $A^T A = AA^T$? Why or why not?

If $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$

So $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$ are both defined since the appropriate dimensions match up.

However, they are not in the same space in general, so they are not equal in general.

Even when $A \in \mathbb{R}^{n \times n}$, $A^T A \neq AA^T$ in general. Why?

Name:________________________________

5. Let $A \in \mathbb{R}^{2\times 2}$ with $a_{11} \neq 0$, and suppose that $\alpha = a_{21}/a_{11}$. Show that A can be factored into a product of the form

$$A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix}.$$

What is the value of b?

$$A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \frac{a_{21}}{\cancel{a_{11}}} \cancel{a_{11}} & \frac{a_{21}}{a_{11}} a_{12} + b \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ if and only if } b = a_{22} - \frac{a_{21} a_{12}}{a_{11}}.$$

6. A matrix is said to be *symmetric* if and only if $A^T = A$.

Let $S = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ be the set of all symmetric 2×2 matrices. Show that S is a subspace of $\mathbb{R}^{2\times 2}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

$A \in S \Rightarrow A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

We need to check S1 and S2.

S1.) If $\alpha \in \mathbb{R}$, then $\alpha A = \alpha \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha d \end{pmatrix} \in S$

since $\alpha a_{12} = \alpha a_{21} = \alpha b$.

S2.) Let $B = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \in S$.

Then $A + B = \begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ b+f & d+g \end{pmatrix} \in S$

since $a_{12} + b_{12} = b + f = a_{21} + b_{21}$. $\square$

Name:________________________________

7. A matrix is said to be *skewsymmetric* if and only if $A^T = -A$.

Show that if $A \in \mathbb{R}^{n\times n}$ is skewsymmetric, then its diagonal entries a_{ii} must be all be 0.

Do the skewsymmetric matrices form a subspace of $\mathbb{R}^{n\times n}$? Justify your answer.

pf] If A is skew symmetric then $a_{ij} = -a_{ji}$ for all $i,j=1,\dots,n$.
In particular, if $i=j$, then $a_{ii}=-a_{ii}$ implies that
$a_{ii}=0$ for all $i=1,\dots,n$. $\square$

Yes, they do form a subspace.

S1.) $A \in S = \{$skew symm. matrices in $\mathbb{R}^{n\times n}\}$.
Then $A=(a_{ij})$ and $a_{ij}=-a_{ji} \Rightarrow \alpha a_{ij} = -\alpha a_{ji}$ for all $\alpha \in \mathbb{R}$. Thus $(\alpha A) \in S$.

S2.) $A, B \in S$, $A=(a_{ij})$, $a_{ij}=-a_{ji}$, $B=(b_{ij})$, $b_{ij}=-b_{ji}$

Then $(a+b)_{ij} = a_{ij}+b_{ij} = -a_{ji}-b_{ji} = -(a_{ji}+b_{ji}) = -(a+b)_{ji}$. $\square$

8. Which of the following subsets are actually subspaces of $\mathbb{R}^2$? Justify your answers.

1. $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$
2. $\{(x_1, x_2) \mid x_1 x_2 = 0\}$
3. $\{(x_1, x_2) \mid x_1 = 3x_2\}$
4. $\{(x_1, x_2) \mid |x_1| = |x_2|\}$
5. $\{(x_1, x_2) \mid x_2 = 2x_1 + 2\}$

Don't turn this part in: Sketch the graph of each subset in the x_1x_2-plane, and compare the graphs to your answers. Do you notice anything interesting?

1. Yes. Verify S1 and S2.

2. No. Let $\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\underline{x}, \underline{y} \in S$, but $\underline{x} + \underline{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S$. So S2 fails.

3. Yes. Verify S1 and S2.

4. No. Let $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $\underline{x}, \underline{y} \in S$, but $\underline{x} + \underline{y} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \notin S$. So S2 fails again.

5. No. Both S1 and S2 fail.

Ex. S1: $\underline{x} = \begin{pmatrix} a \\ 2a+2 \end{pmatrix}$ $\alpha \in \mathbb{R}$

$$\alpha \underline{x} = \begin{pmatrix} \alpha a \\ \alpha(2a+2) \end{pmatrix} = \begin{pmatrix} \alpha a \\ 2\alpha a + 2\alpha \end{pmatrix} \neq \begin{pmatrix} \alpha a \\ 2\alpha a + 2 \end{pmatrix}.$$