Math 511: Linear Algebra
Final Exam, Part II

Friday, 25 July 2014

Name: Key

Instructions: Complete all 3 problems in part I, and 3 of the 4 problems in part II. Clearly mark the problem in part II that you would like to omit. Each problem in part I is worth 20 points; each completed problem in part II is worth 15 points.

Show *enough* work, and follow all instructions carefully. Write your name on each page.

You may *not* use a calculator, or any other electronic device. You may use only a 3×5 index card of your own notes, a pencil, and your brain.

Good Luck!

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Part I. Complete all 3 problems in the space provided. Show enough work. Each problem is worth 20 points.

1. Let S denote the subset of $\mathbb{R}^{2\times 2}$ consisting of all symmetric matrices. [Recall that $A \in \mathbb{R}^{2\times 2}$ is symmetric if and only if $A^T = A$.]

(a.) Show that S is a subspace of $\mathbb{R}^{2\times 2}$.

$A \in S$ looks like $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Similarly, $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$. Let $\alpha \in \mathbb{R}$.

S1. $(\alpha A) = \alpha \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha c \end{pmatrix}$. $(\alpha A)^T = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha c \end{pmatrix}$. Thus $(\alpha A)^T = \alpha A$, and $\alpha A \in S$, whenever $A \in S, \alpha \in \mathbb{R}$.

S2. $(A+B) = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$. $(A+B)^T = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$

Thus $(A+B)^T = (A+B)$ and $A+B \in S$ whenever $A, B \in S$. $\square$

(b.) Find a basis for S. What is the dimension of S?

$\boxed{A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$ form a basis for S.

Indeed any $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = aA_1 + bA_2 + cA_3$.

Clearly A_1, A_2, A_3 are linearly independent.

Thus,

$\boxed{\dim(S) = 3.}$

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2. Let $A \in \mathbb{R}^{n\times n}$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that if $A\mathbf{x} = A\mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$, then A is singular.

Pf| Suppose A is non singular.
Then A^{-1} exists and

$$A\underline{x} = A\underline{y} \Rightarrow A^{-1}(A\underline{x}) = A^{-1}(A\underline{y})$$
$$\Rightarrow (A^{-1}A)\underline{x} = (A^{-1}A)\underline{y}$$
$$\Rightarrow I\underline{x} = I\underline{y}$$
$$\Rightarrow \underline{x} = \underline{y}$$

but this contradicts the assumption that $\underline{x} \neq \underline{y}$! $\times$

Therefore A must be singular. $\square$

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3. Let S be the subspace of $C(-\infty,\infty)$ spanned by $e^x \sin x$ and $e^x \cos x$, and let $D: C(-\infty,\infty) \to C(-\infty,\infty)$ be the differentiation operater: $D(f(x)) = f'(x)$.

(*a.*) Find the matrix representing D with respect to the ordered basis $\{e^x \cos x, e^x \sin x\}$.

$$D(e^x \cos x) = e^x \cos x - e^x \sin x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$D(e^x \sin x) = e^x \cos x + e^x \sin x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so $\boxed{D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}$

(*b.*) Find the inverse matrix D^{-1} and use it to calculate the integral

$$\int e^x \sin x - 2e^x \cos x \, dx.$$

Give your answer as a function in S.

$$D^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$e^x \sin x - 2e^x \cos x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ in coords.

Thus $\int e^x \sin x - 2e^x \cos x \, dx$ is represented by $D^{-1}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$D^{-1}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -1/2 \end{pmatrix}$$ in coords.

Un-packing:

$$\boxed{\int e^x \sin x - 2e^x \cos x \, dx = -\frac{3}{2} e^x \cos x - \frac{1}{2} e^x \sin x}$$

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Part II. Complete 3 of the 4 problems. Show enough work. Clearly mark the one problem that you wish to omit. Each completed problem is worth 15 points.

4. Show that any finite set of vectors in a vector space V that contains the zero vector is linearly dependent.

Let $S = \{\underline{0}, \underline{x}_2, \underline{x}_3, \ldots, \underline{x}_n\}$.

Then $1 \cdot \underline{0} + 0 \cdot \underline{x}_2 + 0 \cdot \underline{x}_3 + \cdots + 0 \cdot \underline{x}_n = \underline{0}$

nontrivial!

So $c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{e}_1$ is a soln of

$A\underline{c} = \underline{0}$ where $A = (\underline{0}, \underline{x}_1, \ldots, \underline{x}_n)$.

Thus the vectors in S are linearly dependent. $\square$

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5. (*a*.) Let $A \in \mathbb{R}^{m\times n}$. State the Rank-Nullity Theorem.

$\text{rank}(A) + \text{null}(A) = n$.

(*b*.) Find bases for the row, column, and null spaces of A. Clearly label each one.

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ 0 & 5 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ 0 & 3 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 2 \end{pmatrix}$$

Thus, $\boxed{\text{Row}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}}$

$\boxed{\text{Col}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}}$

$N(A)$:
$x_1 = 8\alpha$
$x_2 = -2\alpha$
$x_3 = \alpha$

So $\boxed{N(A) = \text{span}\left\{\begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix}\right\}}$

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6. Consider the ordered bases $E = \{1, x, x^2\}$ and $V = \{1, (x+2), (x+2)^2\}$ of P_3.

(a.) Find the transition matrix from E to V.

$$V = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \; : V \to E$$

$$V^{-1}: \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so $\boxed{V^{-1} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}} : E \to V$

(b.) Write the vector $p(x) = 1 + x + x^2$ in both E-coordinates and V-coordinates. Clearly label each answer.

$p(x) = 1 + x + x^2$

$$\boxed{[p]_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_E}$$

$$[p]_V = V^{-1}[p]_E = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_E = \boxed{\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}_V = [p]_V}$$

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7. Consider the following vectors in $\mathbb{R}^2$.

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad \mathbf{y}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{y}_2 = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$

Let $A = (\mathbf{x}_1, \mathbf{x}_2)$ and define a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\mathbf{x}) = A\mathbf{x}$. Find the matrix representing L with respect to the basis $Y = \{\mathbf{y}_1, \mathbf{y}_2\}$.

$A = (\underline{x}_1, \underline{x}_2) = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

$$Y \xrightarrow{Y} E \xrightarrow{L=A} E \xrightarrow{Y^{-1}} Y$$

so $L(\underline{x}) = \underbrace{Y^{-1}AY}_{B}\,\underline{x}$

$Y = (\underline{y}_1, \underline{y}_2) = \begin{pmatrix} 4 & -6 \\ 1 & -2 \end{pmatrix}$

$Y^{-1} = \frac{1}{-2}\begin{pmatrix} -2 & 6 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ \frac{1}{2} & -2 \end{pmatrix}$

so the matrix representing L wrt Y is given by

$$B = \begin{pmatrix} 1 & -3 \\ \frac{1}{2} & -2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4 & -6 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ \frac{1}{2} & -2 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 5 & -6 \end{pmatrix} = \boxed{\begin{pmatrix} -17 & 20 \\ -11 & 13 \end{pmatrix} = B}$$

Thus $L[x]_Y = B[x]_Y$.

Bonus. [5 points] What is the most interesting thing that you learned this summer?

Test cricket favors the batsmen while T20 favors the bowlers (and the viewers).

Long live the bowlers!